CAUCHY RESIDUE THEOREM

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The Cauchy residue theorem, proved in Saff and Snider section 6.1, is one of the central results in complex variable theory. It gives a formula for evaluating an integral around a closed contour C of a function $f(z)$ with a finite number of isolated singularities. The theorem states:

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}(z_j) \tag{1}$$

where the function's singularities are at points z_j where z_j is inside the contour C. Singularities outside C do not contribute to the sum.

Examples of the theorem thus reduce to determining which singularities lie within the contour, and then finding the residue at each such singularity. We have three main methods of finding the residue at a singularity z_0.

(1) Calculate the Laurent series at z_0 and find the coefficient a_{-1} of the $1/(z-z_0)$ term in the series.

(2) If $f(z)$ has a pole of order m at z_0, then

$$\text{Res}(f;z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] \tag{2}$$

(3) If $f(z)$ is a ratio of two functions $P(z)$ and $Q(z)$, and has a simple pole (of order 1) at z_0 (that is, $Q(z_0)=0$ is a simple zero), then

$$\text{Res}(z_0) = \frac{P(z_0)}{Q'(z_0)} \tag{3}$$

We present some examples of the residue theorem where the contour C is a circle centred at the origin.

Example 1. Find

$$\oint_{|z|=5} \frac{\sin z}{z^2-4}\,dz \tag{4}$$

The function has simple poles at $z=\pm 2$, both of which lie inside the contour, so we can use 3 to find

$$\text{Res}(2) = \frac{\sin 2}{2 \times 2} = \frac{\sin 2}{4} \tag{5}$$

$$\text{Res}(-2) = \frac{\sin(-2)}{2 \times (-2)} = \frac{\sin 2}{4} \tag{6}$$

Therefore

$$\oint_{|z|=5} \frac{\sin z}{z^2 - 4} dz = 2\pi i \left(\frac{\sin 2}{4} + \frac{\sin 2}{4} \right) \tag{7}$$

$$= \pi i \sin 2 \tag{8}$$

Example 2. Find

$$\oint_{|z|=3} \frac{e^z}{z(z-2)^3} dz \tag{9}$$

We have a simple pole at $z = 0$ and a pole of order 3 at $z = 2$, both of which lie inside the contour. We can use 2 to find (using Maple to do the derivative)

$$\text{Res}(0) = \lim_{z \to 0} \left[z \frac{e^z}{z(z-2)^3} \right] = -\frac{1}{8} \tag{10}$$

$$\text{Res}(2) = \lim_{z \to 2} \left[\frac{1}{2!} \frac{d^2}{dz^2} \left((z-2)^3 \frac{e^z}{z(z-2)^3} \right) \right] = \frac{e^2}{8} \tag{11}$$

Therefore

$$\oint_{|z|=3} \frac{e^z}{z(z-2)^3} dz = 2\pi i \left(-\frac{1}{8} + \frac{e^2}{8} \right) \tag{12}$$

$$= \frac{\pi i}{4} \left(e^2 - 1 \right) \tag{13}$$

Example 3. Find

$$\oint_{|z|=2\pi} \tan z \, dz \tag{14}$$

We have simple poles whenever $\cos z = 0$, which occurs at $z = \left(n + \frac{1}{2}\right)\pi$. We've seen earlier (Example 7 here) that the residue of $\tan z$ at all these points is -1. The given contour $|z| = 2\pi$ contains 4 singularities, at $z_0 = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, so we have

$$\oint_{|z|=2\pi} \tan z\, dz = 2\pi i\,(4 \times (-1)) = -8\pi i \tag{15}$$

Example 4. Find

$$\oint_{|z|=3} \frac{e^{iz}}{z^2\,(z-2)\,(z+5i)} dz \tag{16}$$

The only poles that lie within the contour are at $z_0 = 0$ and $z_0 = 2$. Using 2 we have, using Maple to do the calculations:

$$\mathrm{Res}\,(0) = \lim_{z\to 0}\left[\frac{d}{dz}\left(z^2 \frac{e^{iz}}{z^2\,(z-2)\,(z+5i)}\right)\right] \tag{17}$$

$$= \frac{2}{25} - \frac{i}{20} \tag{18}$$

$$\mathrm{Res}\,(2) = \lim_{z\to 2}\left[(z-2)\,\frac{e^{iz}}{z^2\,(z-2)\,(z+5i)}\right] \tag{19}$$

$$= \left(\frac{1}{58} + \frac{5i}{116}\right) e^{2i} \tag{20}$$

Therefore

$$\oint_{|z|=3} \frac{e^{iz}}{z^2\,(z-2)\,(z+5i)} dz = 2\pi i\left[\frac{2}{25} - \frac{i}{20} + \left(\frac{1}{58} + \frac{5i}{116}\right) e^{2i}\right] \tag{21}$$

$$= \left(-\frac{5}{58} + \frac{i}{29}\right)\pi e^{2i} + \left(\frac{1}{10} + \frac{4i}{25}\right)\pi \tag{22}$$

where I used Maple to get the last line.

Example 5. Find

$$\oint_{|z|=1} \frac{1}{z^2 \sin z} dz \tag{23}$$

There is only one singularity at $z_0 = 0$ that lies inside the contour. The series for $1/\sin z$ can be found using the same technique as Example 3 here. That is

$$\frac{1}{\sin z} = \sum_{k=0}^{\infty} a_k z^k \tag{24}$$

so that, if we start the series for $1/\sin z$ at the z^{-1} term:

$$\frac{1}{\sin z}\sin z = \left(\sum_{k=-1}^{\infty} a_k z^k\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) \tag{25}$$

$$= a_{-1} + a_0 z + \left(-\frac{a_{-1}}{3!} + a_1\right) z^2 + \ldots \tag{26}$$

This must equal 1, so

$$\begin{aligned} a_{-1} &= 1 \\ a_0 &= 0 \\ a_1 &= \frac{a_{-1}}{3!} = \frac{1}{6} \\ &\vdots \end{aligned} \tag{27}$$

so

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6}z + \ldots \tag{28}$$

This agrees with the standard series expansion for $\frac{1}{\sin z} = \csc z$, which is

$$\csc z = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \mathcal{O}\left(z^7\right) \tag{29}$$

Thus the Laurent series for $\frac{1}{z^2 \sin z}$ begins

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3} + \frac{1}{6z} + \ldots \tag{30}$$

The residue is the coefficient of $1/z$ so

$$\text{Res}\,(0) = \frac{1}{6} \tag{31}$$

and

$$\oint_{|z|=1} \frac{1}{z^2 \sin z} dz = \frac{2\pi i}{6} = \frac{\pi i}{3} \tag{32}$$

Example 6. Find

$$\oint_{|z|=3} \frac{3z+2}{z^4+1} dz \tag{33}$$

This has 4 simple poles at the fourth roots of -1. These are

$$\sqrt[4]{-1} = \begin{cases} e^{\pi i/4} \\ e^{3\pi i/4} \\ e^{-\pi i/4} \\ e^{-3\pi i/4} \end{cases} \tag{34}$$

All four singularities lie on the unit circle, so they all lie within the contour. I couldn't find any simple way of doing this, other than the brute force method of using 3, and letting Maple simplify the results. This gives

$$\begin{aligned} \text{Res}\left(e^{\pi i/4}\right) &= -\frac{3i}{4} - \frac{\sqrt{2}}{4} - \frac{i\sqrt{2}}{4} \\ \text{Res}\left(e^{3\pi i/4}\right) &= \frac{3i}{4} + \frac{\sqrt{2}}{4} - \frac{i\sqrt{2}}{4} \\ \text{Res}\left(e^{-\pi i/4}\right) &= \frac{3i}{4} - \frac{\sqrt{2}}{4} + \frac{i\sqrt{2}}{4} \\ \text{Res}\left(e^{-3\pi i/4}\right) &= -\frac{3i}{4} + \frac{\sqrt{2}}{4} + \frac{i\sqrt{2}}{4} \end{aligned} \tag{35}$$

We can see from this that adding up all 4 residues gives 0, so the integral is zero. I can't help feeling that there is some reason that it should be obvious that the result is zero, but I can't see it. Comments welcome.

Example 7. Find

$$\oint_{|z|=8} \frac{1}{z^2+z+1} dz \tag{36}$$

The quadratic has roots at

$$z_0 = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \tag{37}$$

This gives 2 simple poles, so we can use 3 to get

$$\begin{aligned}\text{Res}\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) &= \frac{1}{2z_0+1} = -\frac{\sqrt{3}i}{3} \\ \text{Res}\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right) &= \frac{\sqrt{3}i}{3}\end{aligned} \tag{38}$$

Thus the sum of the residues is zero, so the integral is zero.

Example 8. Find

$$\oint_{|z|=1} e^{1/z}\sin\frac{1}{z}dz \tag{39}$$

There is an essential singularity at $z=0$. The easiest way of solving this is to expand the two factors in a series.

$$e^{1/z}\sin\frac{1}{z} = \left[1+\frac{1}{z}+\frac{1}{2!z^2}+\ldots\right]\left[\frac{1}{z}-\frac{1}{3!z^3}+\ldots\right] \tag{40}$$

$$= \frac{1}{z}+\frac{1}{z^2}+\ldots \tag{41}$$

We need only the coefficient of $1/z$, which is 1, so $\text{Res}(0)=1$ and

$$\oint_{|z|=1} e^{1/z}\sin\frac{1}{z}dz = 2\pi i \tag{42}$$

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