CAUCHY'S INTEGRAL FORMULA

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Cauchy's integral theorem tells us that the loop integral of an analytic function in a simply connected domain is always zero. Cauchy's integral *formula* is an extension of the theorem, and can be stated as

Theorem 1. *Let D be a simply connected domain containing a simple, closed, positively directed (that is, counterclockwise) contour Γ. If $f(z)$ is analytic within D and z_0 is any point inside Γ, then*

$$f(z_0)=\frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z-z_0}dz \tag{1}$$

The proof is given in Saff and Snider's book, section 4.5.

The theorem can actually be generalized to include derivatives of f, giving the generalized Cauchy integral formula, which is

Theorem 2. *If $f(z)$ is analytic inside and on the same contour Γ referenced in Theorem 1 and z_0 is any point inside Γ, then the nth derivative $f^{(n)}(z_0)$ is given by*

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_\Gamma\frac{f(z)}{(z-z_0)^{n+1}}dz \tag{2}$$

Note that this formula includes 1 when $n=0$.

In practice, these formulas are used to find the integrals on the RHS rather than the simple function values and derivatives on the LHS. This is one area of complex variable theory where we really do seem to get 'something for nothing', or at least something for very little effort.

In the examples that follow, we take Γ to be the circle $|z|=2$ traversed in the positive sense.

Example 1. We have

$$\int_\Gamma\frac{\sin 3z}{z-\frac{\pi}{2}}dz=2\pi i\sin\frac{3\pi}{2}=-2\pi i \tag{3}$$

This follows because $z_0=\frac{\pi}{2}$ is inside the circle.

Example 2. We have

$$\int_\Gamma \frac{ze^z}{2z-3}dz = \frac{1}{2}\int_\Gamma \frac{ze^z}{z-\frac{3}{2}}dz \tag{4}$$

$$= \frac{1}{2}2\pi i\left(\frac{3}{2}e^{3/2}\right) \tag{5}$$

$$= \frac{3\pi i}{2}e^{3/2} \tag{6}$$

Again, $z_0 = \frac{3}{2}$ is inside the circle.

Example 3. We have

$$\int_\Gamma \frac{\cos z}{z^3+9z}dz \tag{7}$$

We can convert this using partial fractions. The denominator factors into

$$z^3+9z = z(z+3i)(z-3i) \tag{8}$$

so we have

$$\frac{1}{z^3+9z} = \frac{A_0^{(1)}}{z} + \frac{A_0^{(2)}}{z+3i} + \frac{A_0^{(3)}}{z-3i} \tag{9}$$

The constants are

$$A_0^{(1)} = \lim_{z\to 0}\frac{z}{z^3+9z} = \frac{1}{9} \tag{10}$$

$$A_0^{(2)} = \lim_{z\to -3i}\frac{z+3i}{z^3+9z} = -\frac{1}{18} \tag{11}$$

$$A_0^{(3)} = \lim_{z\to 3i}\frac{z-3i}{z^3+9z} = -\frac{1}{18} \tag{12}$$

However, observe that $z_0 = \pm 3i$ both lie outside the circle, so the integral depends only on the point $z_0 = 0$, and we have

$$\int_\Gamma \frac{\cos z}{z^3+9z}dz = \int_\Gamma \frac{\cos z}{9z}dz + 0 + 0 \tag{13}$$

$$= 2\pi i\frac{\cos 0}{9} = \frac{2\pi i}{9} \tag{14}$$

Example 4. We have

$$\int_\Gamma \frac{5z^2+2z+1}{(z-i)^3}dz \tag{15}$$

Here, the integral is of the form 2 with $n=2$, so we take the second derivative of the numerator:

$$\frac{d^2}{dz^2}\left(5z^2+2z+1\right)=10 \tag{16}$$

Therefore

$$\int_\Gamma \frac{5z^2+2z+1}{(z-i)^3}dz=\frac{2\pi i}{2!}\times 10=10\pi i \tag{17}$$

Example 5. We have

$$\int_\Gamma \frac{e^{-z}}{(z+1)^2}dz \tag{18}$$

Again, we use 2 with $n=1$, so we have, with $z_0=-1$

$$\int_\Gamma \frac{e^{-z}}{(z+1)^2}dz=\frac{2\pi i}{1!}\left(-e^{-(-1)}\right)=-2\pi ie \tag{19}$$

Example 6. We have

$$\int_\Gamma \frac{\sin z}{z^2(z-4)}dz \tag{20}$$

As in Example 3, we express the denominator using partial fractions. In this case, we have (I used Maple for this, but the calculations are similar to before):

$$\frac{1}{z^2(z-4)}=\frac{1}{16(z-4)}-\frac{1}{4z^2}-\frac{1}{16z} \tag{21}$$

The first term on the RHS has the root $z_0=4$, which lies outside the circle so its integral is zero. Thus we have

$$\int_\Gamma \frac{\sin z}{z^2(z-4)}dz=-\int_\Gamma \frac{\sin z}{4z^2}dz-\int_\Gamma \frac{\sin z}{16z}dz \tag{22}$$

The first term on the RHS gives us, using 2 with $n=1$ and $z_0=0$. We have

$$-\int_\Gamma \frac{\sin z}{4z^2}dz=-\frac{2\pi i}{4}\cos 0=-\frac{\pi}{2}i \tag{23}$$

The second term on the RHS gives

$$-\int_\Gamma \frac{\sin z}{16z}dz=-\frac{2\pi i}{16}\sin 0=0 \tag{24}$$

Thus

$$\int_{\Gamma}\frac{\sin z}{z^{2}\left(z-4\right)}dz=-\frac{\pi}{2}i \tag{25}$$

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