CAUCHY'S INTEGRAL THEOREM

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In Chapter 4 of Saff and Snider's book, they develop a number of theorems that lead up to *Cauchy's integral theorem*, which can be stated as follows.

Theorem 1. *In a simply connected domain (that is, a domain without any holes), an analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals vanish.*

This is actually a special case of the more general theorem, usually called just Cauchy's theorem, which relates a loop integral to the residues of the integrand. More on this in a future post.

Example 1. To illustrate the theorem and some of its exceptions, consider the integral

$$\int_C (z-z_0)^n\,dz \tag{1}$$

where C is a circle of radius r, z_0 is a point inside the circle, and n is an integer (positive or negative).

First, if $n \ge 0$, the function $(z-z_0)^n$ is analytic over the entire plane, so the domain is simply connected and the theorem tells us that the integral is zero. If $n<0$, then the function is analytic only in the punctured plane consisting of all points except z_0, which we can write as $\mathbb{C}\setminus\{z_0\}$. Let us work out the integral explicitly.

We can parametrize the contour as

$$z(t) = z_0 + re^{it} \tag{2}$$

for $0 \le t \le 2\pi$. The integrand is therefore

$$f(z(t)) = (z(t)-z_0)^n \tag{3}$$

$$= r^n e^{int} \tag{4}$$

The integral is then

$$\int_C (z-z_0)^n\,dz = \int_0^{2\pi} f(z(t))\,z'(t)\,dt \tag{5}$$
$$= \int_0^{2\pi} r^n e^{int} i r e^{it} dt \tag{6}$$
$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \tag{7}$$

If $n \neq -1$, the integral is

$$i r^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt = i r^{n+1} \left(\frac{e^{i(n+1)2\pi}}{i(n+1)} - \frac{e^0}{i(n+1)} \right) = 0 \tag{8}$$

If $n = -1$, we have

$$i r^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt = i \int_0^{2\pi} dt = 2\pi i \tag{9}$$

Thus we have the general result

$$\int_C (z-z_0)^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases} \tag{10}$$

The result for $n = -1$ is due to the branch cut in the logarithm. The antiderivative of $(z-z_0)^{-1}$ is $\log(z-z_0)$, which jumps by $2\pi i$ when we cross the branch cut. For all other values of n, the function has an antiderivative everywhere except at z_0 for negative n, so there is no branch cut and the integral is zero.

Example 2. Evaluate

$$\int_\Gamma \frac{2z^2 - z + 1}{(z-1)^2 (z+1)} dz \tag{11}$$

where Γ is the contour given by the lemniscate (figure 8 on its side) in Fig. 1

We begin by expressing the integrand in partial fractions. The decomposition has the form

$$R_{2,3}(z) = \frac{2z^2 - z + 1}{(z-1)^2 (z+1)} = \frac{A_0^{(1)}}{(z-1)^2} + \frac{A_1^{(1)}}{z-1} + \frac{A_0^{(2)}}{z+1} \tag{12}$$

where

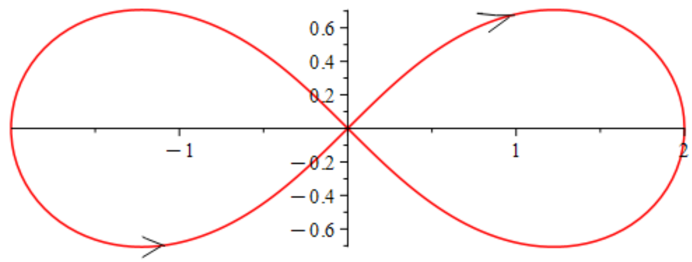

FIGURE 1. Lemniscate contour.

$$A_s^{(j)} = \lim_{z \to \zeta_j} \frac{1}{s!} \frac{d^s}{dz^s} \left[(z - \zeta_j)^{d_j} R_{m,n}(z) \right] \tag{13}$$

and ζ_j are the two roots 1 and -1. Using this formula we have (I used Maple to do the derivative and algebra)

$$A_0^{(1)} = A_1^{(1)} = A_0^{(2)} = 1 \tag{14}$$

Thus we have the contour integral

$$\int_\Gamma \left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{z+1} \right] dz \tag{15}$$

Each lobe of the lemniscate can be continuously deformed into a circle, with the two circles sharing a single point at the origin. The left lobe L contains the point -1 and the right lobe R contains 1. Note, however, that the direction of the left lobe is counterclockwise, while the right lobe is clockwise, which inverts the sign of the integral. From Example 1 above, we have

$$\int_L \frac{1}{z+1} dz = 2\pi i \tag{16}$$

$$\int_R \frac{1}{z-1} dz = -2\pi i \tag{17}$$

All other integrals are zero. In particular

$$\int_L \frac{1}{z-1} dz = \int_R \frac{1}{z+1} dz = \int_\Gamma \frac{1}{(z-1)^2} dz = 0 \tag{18}$$

since the left lobe doesn't contain the point 1, the right lobe doesn't contain the point -1, and the loop integral of $1/(z-1)^2$ is zero for any loop. Thus we have

$$\int_{\Gamma}\frac{2z^{2}-z+1}{(z-1)^{2}(z+1)}dz=2\pi i-2\pi i=0 \tag{19}$$

Example 3. Let

$$I=\oint_{|z|=2}\frac{dz}{z^{2}(z-1)^{3}} \tag{20}$$

That is, we're doing the integral around the circle $|z|=2$.

If $R>2$ the integral

$$I(R)=\oint_{|z|=R}\frac{dz}{z^{2}(z-1)^{3}} \tag{21}$$

has the same value, since the roots of the denominator are 0 and 1, both of which are contained within any circle with radius ≥ 2. We can get an upper bound on $I(R)$ by using the technique of estimating contour integrals. To find an upper bound, we need to find the minimum value of the denominator over the circle $|z|=R$. We have

$$\left|z^{2}\right|=R^{2} \tag{22}$$

everywhere on the circle. Also, the minimum value of $|z-1|$ occurs when $z=R$. The length of the circle is $2\pi R$, so putting it all together, we have

$$|I(R)|\leq\frac{2\pi R}{R^{2}(R-1)^{3}}=\frac{2\pi}{R(R-1)^{3}} \tag{23}$$

If we let R go to infinity we have

$$\lim_{R\to\infty}|I(R)|=\lim_{R\to\infty}\frac{2\pi}{R(R-1)^{3}}=0 \tag{24}$$

Since all circles with $R>2$ give the same value of the integral, we must have

$$I(R)=0 \tag{25}$$

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