CAUCHY-RIEMANN EQUATIONS

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In order for a function of a complex variable to be differentiable, the limit

$$f'(z_0) \equiv \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1}$$

must exist and be unique. That is, it must give the same value at a point z_0 no matter in which direction we approach z_0 when calculating the limit.

We can use this fact to derive a necessary condition for differentiability. If we write $z = x + iy$ and

$$f(z) = u(x,y) + iv(x,y) \tag{2}$$

we can calculate the limit 1 along horizontal and vertical paths and require that the results are equal. For a horizontal path, $\Delta z = \Delta x$ and we have

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \tag{3}$$

Since u and v are real functions, we can write this in terms of the real derivatives of u and v with respect to x.

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \tag{4}$$

For a vertical path, $\Delta z = i\Delta y$ and we have

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \tag{5}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \tag{6}$$

$$= -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \tag{7}$$

Equating the real and imaginary parts of 4 and 7 we have

Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{8}$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \tag{9}$$

These are the Cauchy-Riemann equations in rectangular coordinates. They are a necessary (but not quite sufficient) condition for a function to be differentiable.

Example 1. The function $f(z)=\overline{z}$ is not differentiable. We have

$$f(x)=x-iy \tag{10}$$

so

$$\frac{\partial u}{\partial x}=1 \tag{11}$$

$$\frac{\partial v}{\partial y}=-1 \tag{12}$$

This violates 8 so the function is not differentiable.

Example 2. Cauchy-Riemann equations in polar coordinates. In polar coordinates

$$z=re^{i\theta} \tag{13}$$

We can go through a similar derivation to that above, once we have chosen the two paths along which to evaluate the limits. The most convenient paths are a ray of constant $\theta=\theta_0$, and a circle of constant $r=r_0$. We assume that we have the two real functions $u(r,\theta)$ and $v(r,\theta)$ already written in terms of polar coordinates.

Along the ray, we have $z=re^{i\theta_0}$ with θ_0 constant, so $\Delta z=\Delta re^{i\theta_0}$ and

$$f'(z_0)=\lim_{\Delta r\to 0}\frac{u(r_0+\Delta r,\theta_0)-u(r_0,\theta_0)+iv(r_0+\Delta r,\theta_0)-v(r_0,\theta_0)}{\Delta re^{i\theta_0}} \tag{14}$$

$$=\frac{1}{e^{i\theta_0}}\left(\frac{\partial u}{\partial r}(r_0,\theta_0)+i\frac{\partial v}{\partial r}(r_0,\theta_0)\right) \tag{15}$$

Turning now to the limit along the circle $r=r_0$, we have

$$\Delta z=r_0\left(e^{i(\theta_0+\Delta\theta)}-e^{i\theta_0}\right) \tag{16}$$

To first order in $\Delta\theta$ (all higher orders will go to zero in the limit $\Delta\theta \to 0$), this is

$$\Delta z = r_0 \left(e^{i\theta_0} \left(1 + \left.\frac{de^{i\theta}}{d\theta}\right|_{\theta_0} \Delta\theta - 1 \right) \right) \tag{17}$$

$$= r_0 \left.\frac{de^{i\theta}}{d\theta}\right|_{\theta_0} \Delta\theta \tag{18}$$

$$= r_0 e^{i\theta_0} i\Delta\theta \tag{19}$$

The limit becomes

$$f'(z_0) = \lim_{\Delta\theta\to 0} \frac{u(r_0,\theta_0+\Delta\theta) - u(r_0,\theta_0) + iv(r_0,\theta_0+\Delta\theta) - v(r_0,\theta_0)}{r_0 e^{i\theta_0} i\Delta\theta} \tag{20}$$

$$= \frac{1}{ir_0 e^{i\theta_0}} \left(\frac{\partial u}{\partial\theta}(r_0,\theta_0) + i\frac{\partial v}{\partial\theta}(r_0,\theta_0) \right) \tag{21}$$

Equating real and imaginary parts of 15 and 21 we have

$$\frac{\partial u}{\partial r}(r_0,\theta_0) = \frac{1}{r_0}\frac{\partial v}{\partial\theta}(r_0,\theta_0) \tag{22}$$

$$\frac{\partial v}{\partial r}(r_0,\theta_0) = -\frac{1}{r_0}\frac{\partial u}{\partial\theta}(r_0,\theta_0) \tag{23}$$

These are the Cauchy-Riemann equations in polar form.

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