CIRCLE OF CONVERGENCE FOR POWER SERIES

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A *power series* is a series of the form

$$P=\sum_{k=0}^{\infty}a_{k}\left(z-z_{0}\right)^{k} \tag{1}$$

The constants a_k are called the coefficients of the series.

The convergence of a power series is given by the following theorem.

Theorem 1. *For any power series there is a real number R between 0 and ∞ inclusive which depends only on the coefficients $\{a_k\}$ such that*

(i) the series converges for $|z-z_0|<R$. The circle $|z-z_0|=R$ is called the circle of convergence, and R is the radius of convergence.

(ii) the series converges uniformly in any closed subdisk $|z-z_0|\le R'<R$.

(iii) the series diverges for $|z-z_0|>R$.

The proof of this theorem is quite technical and is given in Saff and Snider, Section 5.4.

Theorem 2. *Given a power series 1, suppose that*

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=L \tag{2}$$

Then the radius of convergence is given by

$$R=\frac{1}{L} \tag{3}$$

Proof. We use the ratio test which states that for a series

$$S=\sum_{k=0}^{\infty}c_{k} \tag{4}$$

if

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|=A \tag{5}$$

then the series converges if $L<1$ and diverges if $L>1$. Applying this to 1 we take

$$c_k = a_k (z - z_0)^k \tag{6}$$

so that

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}(z-z_0)^{k+1}}{a_k (z-z_0)^k}\right| \tag{7}$$

$$= L|z-z_0| \tag{8}$$

Requiring this to be less than 1, we have

$$L|z-z_0| < 1 \tag{9}$$

so

$$|z-z_0| < \frac{1}{L} \tag{10}$$

Thus the radius of convergence is $R=\frac{1}{L}$. □

Here are a few examples of this theorem. Closed formulas for the series were obtained using Maple, where possible.

Example 1. For

$$\sum_{k=0}^{\infty} k^3 z^k \tag{11}$$

we have $z_0=0$ and $a_k=k^3$ so

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)^3}{k^3} \tag{12}$$

$$= \lim_{k\to\infty}\frac{k^3+3k^2+3k+1}{k^3} \tag{13}$$

$$= 1 \tag{14}$$

The circle of convergence is

$$|z| = 1 \tag{15}$$

This series in closed form is (for $0<z<1$):

$$\sum_{k=0}^{\infty} k^3 z^k = \frac{z\left(z^2+4z+1\right)}{(z-1)^4} \tag{16}$$

Example 2. For

$$\sum_{k=0}^{\infty} 2^k (z-1)^k \tag{17}$$

we have $z_0 = 1$ and $a_k = 2^k$ so

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{2^{k+1}}{2^k} = 2 = L \tag{18}$$

The circle of convergence is

$$|z-1| = \frac{1}{L} = \frac{1}{2} \tag{19}$$

This is a geometric series with sum (for $|z-1| < \frac{1}{2}$):

$$\sum_{k=0}^{\infty} 2^k (z-1)^k = \frac{1}{3-2z} \tag{20}$$

Example 3. For

$$\sum_{k=0}^{\infty} k! z^k \tag{21}$$

we have $z_0 = 0$ and $a_k = k!$ so

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{(k+1)!}{k!} \tag{22}$$

$$= \lim_{k\to\infty} (k+1) \tag{23}$$

$$= \infty \tag{24}$$

Thus the circle of convergence is the single point $z = 0$ and the series diverges for any other value of z.

Example 4. For

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{3^k} (z-i)^k \tag{25}$$

we have $z_0 = i$ and $a_k = \frac{(-1)^k k}{3^k}$ so

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+1)}{3^{k+1}}\times\frac{3^k}{(-1)^k k}\right| \tag{26}$$
$$= \lim_{k\to\infty}\frac{k+1}{3k} \tag{27}$$
$$= \frac{1}{3} \tag{28}$$

Thus the circle of convergence is

$$|z-i| = 3 \tag{29}$$

The sum is (for $|z-i|<3$):

$$\sum_{k=0}^{\infty}\frac{(-1)^k k}{3^k}(z-i)^k = \frac{-3z+3i}{(z+3-i)^2} \tag{30}$$

Example 5. For

$$\sum_{k=1}^{\infty}\frac{(3-i)^k}{k^2}(z+2)^k \tag{31}$$

we have $z_0=-2$ and $a_k=(3-i)^k/k^2$ so

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(3-i)^{k+1}}{(k+1)^2}\times\frac{k^2}{(3-i)^k}\right| \tag{32}$$
$$= \lim_{k\to\infty}\left|(3-i)\frac{k^2}{(k+1)^2}\right| \tag{33}$$
$$= \sqrt{10}\lim_{k\to\infty}\left|\frac{1}{1+2/k+1/k^2}\right| \tag{34}$$
$$= \sqrt{10} \tag{35}$$

Thus the circle of convergence is

$$|z+2| = \sqrt{10} \tag{36}$$

Maple couldn't find a closed form for this series.

Example 6. For

$$\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k} \tag{37}$$

In this case, the series is a geometric series of form

$$\sum_{k=0}^{\infty} c^k = \sum_{k=0}^{\infty} \left(\frac{z^2}{4}\right)^k \tag{38}$$

which converges if $|c| < 1$, so we must have

$$\left|\frac{z^2}{4}\right| < 1 \tag{39}$$

or

$$|z| < 2 \tag{40}$$

Thus the circle of convergence is

$$|z| = 2 \tag{41}$$

The series is (for $|z| < 2$):

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{4^k} = \frac{4}{4 - z^2} \tag{42}$$

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