# CIRCULAR ARCS AROUND SIMPLE POLES

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Our earlier examples of improper integrals (with infinite limits on the real axis) used Cauchy's residue theorem, but relied on the singularities of the integrand lying off the real axis. Here, we'll look at a theorem that can be used to calculate improper integrals along the real axis where the integrand has one or more simple poles on the real axis.

In this case, the contour along the real axis is indented by a small semi-circular arc of radius $r$ centred at the pole, and the limit is taken as $r \to 0$.

The main result is a lemma proved in Saff and Snider, Section 6.5. It is

**Lemma 1.** *Let $f(z)$ be a function with a simple pole at $z=c$, and let $T_r$ be a circular arc with centre at $c$ and defined by*

$$T_r: \quad z = c + re^{i\theta} \tag{1}$$

*where $\theta_1 \le \theta \le \theta_2$. Note that the arc does not have to be a complete circle; it can be only an arc that is a portion of a circle. Also note that $T_r$ is traversed counterclockwise, in the usual manner.*

*Then*

$$\lim_{r \to 0^+} \int_{T_r} f(z)\,dz = i(\theta_2 - \theta_1)\,Res(f;c) \tag{2}$$

*In many applications, the arc is traversed in the clockwise direction, so*

$$\lim_{r \to 0^+} \int_{T_r} f(z)\,dz = -i(\theta_2 - \theta_1)\,Res(f;c) \tag{3}$$

Here, we give a few examples of calculating the integral along such arcs, in the limit as $r \to 0$.

**Example 1.** Find

$$\lim_{r \to 0^+} \int_{T_r} \frac{2z^2+1}{z}\,dz \tag{4}$$

with $z = re^{i\theta}$ and $0 \le \theta \le \frac{\pi}{2}$.

There is a pole at $z=0$ which is the centre of the arc so we can apply the lemma. The integrand is

$$\frac{2z^2+1}{z}=\frac{1}{z}+2z \tag{5}$$

The residue is

$$\text{Res}(z=0)=1 \tag{6}$$

so

$$\lim_{r\to 0^+}\int_{T_r}\frac{2z^2+1}{z}dz=\frac{\pi}{2}i \tag{7}$$

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**Example 2.** Find

$$\lim_{r\to 0^+}\int_{T_r}\frac{e^{3iz}}{z^2-1}dz \tag{8}$$

with $z=1+re^{i\theta}$ and $\frac{\pi}{4}\le\theta\le\pi$. We can write the integrand as

$$\frac{e^{3iz}}{z^2-1}=\frac{e^{3iz}}{(z+1)(z-1)} \tag{9}$$

$$=\frac{e^{3iz}}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right) \tag{10}$$

The residue at $z=1$ is

$$\text{Res}(1)=\frac{e^{3i}}{2} \tag{11}$$

so

$$\lim_{r\to 0^+}\int_{T_r}\frac{e^{3iz}}{z^2-1}dz=\frac{3\pi i}{4}\frac{e^{3i}}{2}=\frac{3\pi ie^{3i}}{8} \tag{12}$$

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**Example 3.** Find

$$\lim_{r\to 0^+}\int_{T_r}\frac{\text{Log}z}{z-1}dz \tag{13}$$

with $z=1+re^{-i\theta}$ and $\pi\le\theta\le 2\pi$. In this case, due to the negative exponent in $z=1+re^{-i\theta}$, the arc is traversed in the clockwise direction, so we use 3. However, for the pole at $z=1$, we have

$$\text{Res}(1)=\text{Log}1=0 \tag{14}$$

so the integral is zero.

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**Example 4.** Find

$$\lim_{r\to 0^+}\int_{T_r}\frac{e^z-1}{z^2}dz \tag{15}$$

with $z=re^{-i\theta}$ and $\pi\le\theta\le 2\pi$. Again, we traverse in the clockwise direction. We expand the integrand in a series:

$$\frac{e^z-1}{z^2}=\frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots-1\right) \tag{16}$$

$$=\left(\frac{1}{z}+\frac{1}{2}+\frac{z}{3!}+\ldots\right) \tag{17}$$

There is a simple pole at $z=0$, so

$$\operatorname{Res}(0)=1 \tag{18}$$

and

$$\lim_{r\to 0^+}\int_{T_r}\frac{e^z-1}{z^2}dz=-i\pi \tag{19}$$

## PINGBACKS

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