COMPLEX EXPONENTIAL, TRIG AND HYPERBOLIC FUNCTIONS

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The exponential of a complex number $z=x+iy$, where x and y are real, is given by Euler's equation

$$e^{z}=e^{x}\left(\cos y+i\sin y\right) \tag{1}$$

This gives rise to expressions for the trig functions of a real number in terms of exponentials. We have

$$\begin{aligned}\cos y &= \frac{e^{iy}+e^{-iy}}{2}\\ \sin y &= \frac{e^{iy}-e^{-iy}}{2i}\end{aligned} \tag{2}$$

We can generalize these formulas to give a definition for trig functions of a complex argument. That is, we *define*

$$\begin{aligned}\cos z &\equiv \frac{e^{iz}+e^{-iz}}{2}\\ \sin z &\equiv \frac{e^{iz}-e^{-iz}}{2i}\end{aligned} \tag{3}$$

where z is now any complex number.

We can also generalize the definition of the hyperbolic functions to the complex domain.

$$\begin{aligned}\cosh z &\equiv \frac{e^{z}+e^{-z}}{2}\\ \sinh z &\equiv \frac{e^{z}-e^{-z}}{2}\end{aligned} \tag{4}$$

With these definitions, all the usual identities for trig and hyperbolic functions transfer over to the complex domain. For trig functions, we have

$$\cos^2 z + \sin^2 z = 1 \tag{5}$$
$$\sin(-z) = -\sin z \tag{6}$$
$$\cos(-z) = \cos z \tag{7}$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \tag{8}$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \tag{9}$$
$$\sin 2z = 2 \sin z \cos z \tag{10}$$
$$\cos 2z = \cos^2 z - \sin^2 z \tag{11}$$

A sample of hyperbolic identities is

$$\cosh^2 z - \sinh^2 z = 1 \tag{12}$$
$$\sinh(-z) = -\sinh z \tag{13}$$
$$\cosh(-z) = \cosh z \tag{14}$$
$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \sinh z_2 \cosh z_1 \tag{15}$$
$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2 \tag{16}$$
$$\cosh 2z = \sinh^2 z + \cosh^2 z \tag{17}$$
$$\sinh 2z = 2 \sinh z \cosh z \tag{18}$$

These can all be proved using the definitions 3 and 4.

There are also a couple of identities relating the trig and hyperbolic functions. We have

$$\sin(iz) = \frac{e^{(iz)i} - e^{-(iz)i}}{2i} \tag{19}$$
$$= \frac{e^{-z} - e^{z}}{2i} \tag{20}$$
$$= -\frac{\sinh z}{i} \tag{21}$$
$$\cos(iz) = \frac{e^{(iz)i} + e^{-(iz)i}}{2} \tag{22}$$
$$= \cosh z \tag{23}$$

Note that, for complex numbers, the magnitudes of sine and cosine are no longer restricted to the interval $[0,1]$. For example, since $\sin(iz) = -\sinh z/i = (e^{-z} - e^{z}/2i)$, we can make the magnitude of $\sin(iz)$ as large as we like by choosing a large enough (real) value for z. The identity

$\cos^2 z + \sin^2 z = 1$ is still true, since one of $\cos^2 z$ or $\sin^2 z$ can now be negative.

Example 1. From 8, The RHS is

$$\sin z_1 \cos z_2 + \sin z_2 \cos z_1 = \frac{e^{z_1 i} - e^{-z_1 i}}{2i} \frac{e^{z_2 i} + e^{-z_2 i}}{2} + \frac{e^{z_2 i} - e^{-z_2 i}}{2i} \frac{e^{z_1 i} + e^{-z_1 i}}{2} \tag{24}$$

Multiplying this out, we find, taking the two terms on the RHS separately:

$$\frac{e^{z_1 i} - e^{-z_1 i}}{2i} \frac{e^{z_2 i} + e^{-z_2 i}}{2} = \frac{e^{(z_2+z_1)i} - e^{-(z_2+z_1)i}}{4i} + \frac{e^{(-z_2+z_1)i} - e^{-(-z_2+z_1)i}}{4i} \tag{25}$$

$$= \frac{\sin(z_2+z_1)}{2} + \frac{\sin(-z_2+z_1)}{2} \tag{26}$$

$$\frac{e^{z_2 i} - e^{-z_2 i}}{2i} \frac{e^{z_1 i} + e^{-z_1 i}}{2} = \frac{\sin(z_2+z_1)}{2} - \frac{\sin(-z_2+z_1)}{2} \tag{27}$$

Combining the terms, we have

$$\frac{e^{z_1 i} - e^{-z_1 i}}{2i} \frac{e^{z_2 i} + e^{-z_2 i}}{2} + \frac{e^{z_2 i} - e^{-z_2 i}}{2i} \frac{e^{z_1 i} + e^{-z_1 i}}{2} = \sin(z_1 + z_2) \tag{28}$$

Example 2. Another identity is

$$\sin z_2 - \sin z_1 = 2\cos\frac{z_2+z_1}{2}\sin\frac{z_2-z_1}{2} \tag{29}$$

This can be verified by expanding the RHS using 8 and 9. Using Maple to simplify things, we get, using the shorthand notation

$$\begin{aligned} s_k &\equiv \sin\frac{z_k}{2} \\ c_k &\equiv \cos\frac{z_k}{2} \end{aligned} \tag{30}$$

$$2\cos\frac{z_2+z_1}{2}\sin\frac{z_2-z_1}{2} = 2c_2c_1^2s_2 - 2c_2^2c_1s_1 - 2s_2^2s_1c_1 + 2s_2s_1^2c_2 \tag{31}$$

$$= 2c_2s_2\left(c_1^2+s_1^2\right) - 2c_1s_1\left(c_2^2+s_2^2\right) \tag{32}$$

$$= 2s_2c_2 - 2s_1c_1 \tag{33}$$

$$= \sin z_2 - \sin z_1 \tag{34}$$

where we used 10 to get the last line.

We now do a few examples to show how these functions can be reduced to the standard form $x + iy$.

Example 3. We have

$$\exp\left(2+\frac{\pi i}{4}\right) = e^2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) \tag{35}$$

$$= e^2\frac{\sqrt{2}}{2}(1+i) \tag{36}$$

Example 4. We have

$$\frac{e^{1+i3\pi}}{e^{-1+i\pi/2}} = e^2 e^{i5\pi/2} \tag{37}$$

$$= e^2 e^{\pi/2} \tag{38}$$

$$= e^2 i \tag{39}$$

Example 5. We have

$$\sin(2i) = \frac{e^{(2i)i}-e^{-(2i)i}}{2i} \tag{40}$$

$$= \frac{e^{-2}-e^{2}}{2i} \tag{41}$$

$$= i\sinh 2 \tag{42}$$

This is an example of identity 21

Example 6. We have

$$\cos(1-i) = \cos 1\cos(-i)+\sin 1\sin(-i) \tag{43}$$

$$= \cos 1\frac{e^{(-i)i}+e^{-(-i)i}}{2}+\sin 1\frac{e^{(-i)i}-e^{-(-i)i}}{2i} \tag{44}$$

$$= \cos 1\cosh 1+i\sin 1\sinh 1 \tag{45}$$

This uses 21 and 23.

Example 7. We have

$$\sinh(1+i\pi) = \sinh 1\cosh i\pi+\sinh i\pi\cosh 1 \tag{46}$$

$$= \sinh 1\frac{e^{i\pi}+e^{-i\pi}}{2}+\frac{e^{i\pi}-e^{-i\pi}}{2}\cosh 1 \tag{47}$$

$$= \sinh 1\cos\pi+i\sin\pi\cosh 1 \tag{48}$$

$$= -\sinh 1 \tag{49}$$

Example 8. We have, using 23

$$\cosh\frac{i\pi}{2}=\cos\left(i\frac{i\pi}{2}\right)=\cos\left(-\frac{\pi}{2}\right)=0 \tag{50}$$

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