COMPLEX EXPONENTIALS AND EULER'S EQUATION

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A complex number z may also be written as an exponential. To see the motivation for this, we propose that some of the properties of the exponential function carry over to the case where the exponent is complex. That is

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{1}$$

If $z = x + iy$, then we have

$$e^{z} = e^{x}e^{iy} \tag{2}$$

The e^{x} factor is a real number, but we need to provide an interpretation for the exponential of an imaginary number. We try taking the derivative:

$$\frac{d}{dy}e^{iy} = ie^{iy} \tag{3}$$

$$\frac{d^2}{dy^2}e^{iy} = i^2e^{iy} = -e^{iy} \tag{4}$$

The second derivative has the form

$$\frac{d^2f}{dy^2} = -f \tag{5}$$

which has the general solution

$$f(y) = e^{iy} = A\cos y + B\sin y \tag{6}$$

We can find A and B from the initial conditions. We require $f(0) = e^{0} = 1$, from which

$$A\cos 0 + B\sin 0 = 1 \tag{7}$$

so

$$A = 1 \tag{8}$$

The first derivative gives us

$$f'(0) = ie^0 = i \tag{9}$$

giving us

$$-A\sin 0 + B\cos 0 = i \tag{10}$$

so

$$B = i \tag{11}$$

Thus we get the general form

Euler's equation.

$$\boxed{e^z = e^x(\cos y + i\sin y)} \tag{12}$$

This is known as *Euler's equation*.

Example 1. Quotient of exponentials.

$$\begin{aligned}
\frac{e^{1+3\pi i}}{e^{-1+\pi i/2}} &= e^{1+3\pi i}e^{1-\pi i/2} && (13)\\
&= e^2 e^{5\pi i/2} && (14)\\
&= e^2\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) && (15)\\
&= ie^2 && (16)
\end{aligned}$$

Example 2. Exponential of an exponential.

$$\begin{aligned}
\exp\left(4e^{i\pi/3}\right) &= \exp\left(4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right) && (17)\\
&= e^{2+2\sqrt{3}i} && (18)\\
&= e^2\left(\cos\left(2\sqrt{3}\right) + i\sin\left(2\sqrt{3}\right)\right) && (19)
\end{aligned}$$

Example 3. Powers of complex numbers are often easier to calculate using exponentials.

$$\begin{aligned}
(1+i)^6 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^6 && (20)\\
&= \left(\sqrt{2}e^{i\pi/4}\right)^6 && (21)\\
&= 2^3 e^{3\pi i/2} && (22)\\
&= -8i && (23)
\end{aligned}$$

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