COMPLEX LOGARITHMS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 27 December 2024.

The logarithm of a complex number is defined starting from the polar form

$$z = re^{i\theta} \tag{1}$$

In this form, $r = |z|$ is unique but θ can take on an infinite number of values by adding multiples of 2π. If we use the principal value of the argument, denoted by Arg z, then θ lies in the interval $(-\pi, \pi]$. Using this argument, we can define the *principal value of the logarithm* Log z to be

$$\boxed{\text{Log } z \equiv \text{Log}\,|z| + i\text{Arg } z} \tag{2}$$

We use a capital L to denote the principal logarithm.[1]

A more general form of the logarithm $\log z$, with a small ℓ, is defined to be one in which the argument can take on any of its infinite number of values. That is

$$\begin{aligned} \log z &\equiv \text{Log } |z| + i \arg z & (3)\\ &= \text{Log } |z| + i\text{Arg } z + 2k\pi i & (4) \end{aligned}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

Example 1. We have

$$\begin{aligned} \log i &= \text{Log } |i| + i \arg i & (5)\\ &= \text{Log } 1 + i\frac{\pi}{2} + 2k\pi i & (6)\\ &= i\frac{\pi}{2} + 2k\pi i & (7) \end{aligned}$$

Example 2. We have

$$\log(1-i) = \text{Log } \sqrt{2} + i \arg(1-i) \tag{8}$$

[1] All logarithms in these notes are *natural* logarithms, that is, to base e. I'm following the notation in Saff and Snider's book, where they use Log and log in place of the more common ln used with real variables.

The principal argument is

$$\text{Arg }(1-i) = -\frac{\pi}{4} \tag{9}$$

and

$$\text{Log }\sqrt{2} = \frac{1}{2}\text{Log }2 \tag{10}$$

so

$$\log(1-i) = \frac{1}{2}\text{Log }2 - i\frac{\pi}{4} + 2k\pi i \tag{11}$$

Example 3. We have

$$\text{Log}(-i) = \text{Log}|-i| + i\text{Arg}(-i) \tag{12}$$
$$= 0 - i\frac{\pi}{2} \tag{13}$$
$$= -i\frac{\pi}{2} \tag{14}$$

In this case, we're looking only for the principal logarithm, so only one value of the argument is required.

Example 4. We have

$$\text{Log}\left(\sqrt{3}+i\right) = \text{Log }2 + i\text{Arg}\left(\sqrt{3}+i\right) \tag{15}$$

The principal argument is

$$\text{Arg}\left(\sqrt{3}+i\right) = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6} \tag{16}$$

so

$$\text{Log}\left(\sqrt{3}+i\right) = \text{Log }2 + i\frac{\pi}{6} \tag{17}$$

Example 5. The rules for the logarithms of products and quotients are the same as those for real numbers, *provided* we are dealing with the general logarithm (with a small ℓ). These rules are not always valid for the principal logarithms, as this example shows.

We have

$$\text{Log }i = i\frac{\pi}{2} \tag{18}$$
$$\text{Log }(i-1) = \frac{1}{2}\text{Log }2 + i\frac{3\pi}{4} \tag{19}$$

The Log of the product is

$$\text{Log }[i(i-1)] = \text{Log }(-1-i) \tag{20}$$

$$= \frac{1}{2}\text{Log }2 - i\frac{3\pi}{4} \tag{21}$$

since $-1-i$ is in the third quadrant.

Adding together 18 and 19, however, gives

$$\text{Log }i + \text{Log }(i-1) = \frac{1}{2}\text{Log }2 + i\frac{5\pi}{4} \tag{22}$$

The argument of $5\pi/4$ does not lie in the interval $(-\pi,\pi]$ so is not a valid quantity for a principal argument. It *is* valid for a general argument, as it is $-3\pi/4+2\pi$.

PINGBACKS

Pingback: Branches of logarithms
Pingback: Boundary value problems - circles
Pingback: Complex powers
Pingback: Inverse trigonometric functions
Pingback: Analytic branches of roots
Pingback: Inverse secant
Pingback: Estimating contour integrals