COMPLEX MATRIX PRODUCTS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 6 November 2024.

Suppose we have an $n\times n$ matrix A with complex entries $[a_{ij}]$ and a column vector u, also with complex entries $[u_k]$. The *hermitian conjugate* $u^{\dagger}$ of a vector or matrix is the complex conjugate transpose. That is, we swap rows and columns and take the complex conjugate of all entries.

Theorem 1. *For the matrix A and vector u, if $u^{\dagger}Au=0$ for all complex vectors u, then $A=[0]$, that is, all entries a_{ij} of A are zero.*

Proof. This is easiest to see if we consider A to be a fixed, small size such as 3×3. We'll consider first the case where

$$u=\begin{bmatrix} u_1\\ u_2\\ 0\end{bmatrix} \tag{1}$$

$$u^{\dagger}=\begin{bmatrix} \overline{u}_1 & \overline{u}_2 & 0\end{bmatrix} \tag{2}$$

and

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{3}$$

Then

$$u^{\dagger}A=\begin{bmatrix} (\overline{u}_1a_{11}+\overline{u}_2a_{21}) & (\overline{u}_1a_{12}+\overline{u}_2a_{22}) & (\overline{u}_1a_{13}+\overline{u}_2a_{23})\end{bmatrix} \tag{4}$$

The total product is then

$$u^{\dagger}Au=u_1\overline{u}_1a_{11}+u_1\overline{u}_2a_{21}+u_2\overline{u}_1a_{12}+u_2\overline{u}_2a_{22} \tag{5}$$

$$=|u_1|^2a_{11}+|u_2|^2a_{22}+u_1\overline{u}_2a_{21}+u_2\overline{u}_1a_{12} \tag{6}$$

Since we require $u^{\dagger}Au=0$ for *all* complex vectors u, we can make a few specific choices for u. First, suppose $u_1=1$ and $u_2=0$. Then from 6 we have

$$u^{\dagger}Au=|u_1|^2a_{11}=a_{11}=0 \tag{7}$$

Similarly, by choosing $u_1=0$ and $u_2=1$, we find that

$$a_{22}=0 \tag{8}$$

We could generalize the argument above to an $n\times n$ matrix of any size, and by choosing u with $u_i=1$ and all other entries zero, we find that all diagonal elements $a_{ii}=0$.

Returning to the 3×3 case, we can therefore write 6 as

$$u^{\dagger}Au=u_1\overline{u}_2a_{21}+u_2\overline{u}_1a_{12} \tag{9}$$

Now take $u_1=u_2=1$. This gives us

$$u^{\dagger}Au=a_{21}+a_{12}=0 \tag{10}$$

or

$$a_{12}=-a_{21} \tag{11}$$

Finally, we can take $u_1=1$ and $u_2=i$. Then we have

$$u^{\dagger}Au=-ia_{21}+ia_{12}=0 \tag{12}$$

or

$$a_{12}=a_{21} \tag{13}$$

The only way that both 11 and 13 can be true at the same time is if

$$a_{12}=a_{21}=0 \tag{14}$$

Again, we can generalize this argument to an $n\times n$ matrix of any size, and choose the entries of u in pairs as above. Thus we find that all entries of A must be zero. □

Corollary 1. *If we require $u^{\dagger}Au=0$ only for all vectors u with real entries, then the above theorem is not true.*

Proof. The proof of the theorem relied on the fact that we could choose a vector u with complex entries, in order to derive the condition 13. If u is restricted to real entries, then we can derive only condition 11, so all we can say is that, if u contains only real entries, then $u^{\dagger}Au=0$ for some vectors u if A is antisymmetric. □

PINGBACKS

Pingback: Hermitian matrix theorems