COMPLEX NUMBERS IN POLAR FORM

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A complex number z can be written as $z = x + iy$ where x is the real part and y is the imaginary part. This is the *rectangular* form for z. If we plot z on the complex plane and draw a vector from the origin to z, then we can define the length r of the vector and the angle θ that the vector makes with the real (x) axis. From trigonometry, the relations between x and y and r and θ are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{1}$$

The inverse relations are

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned} \tag{2}$$

There is an inherent ambiguity in the definition of θ depending on how we choose to define angles in the complex plane. The standard definition takes $-\pi < \theta \le \pi$. With this definition, θ is called the principal argument of z, and we write

$$\theta = \text{Arg} z \tag{3}$$

with a capital 'A'. Other possibilities involve adding a multiple of 2π to Arg$z$. Such values for θ are usually written as $\arg z$, with a lowercase 'a'.

From these definitions we can write the *polar* form of a complex number as

Readers may also be familiar with the exponential form $z = re^{i\theta}$, which we'll get to in a future post.

$$z = r(\cos\theta + i\sin\theta) \equiv r\text{cis}\theta \tag{4}$$

The product of two complex numbers in polar form is found as follows.

$$z_1 z_2 = r_1 (\cos\theta_1 + i\sin\theta_1) r_2 (\cos\theta_2 + i\sin\theta_2) \tag{5}$$
$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)) \tag{6}$$
$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \tag{7}$$
$$= r_1 r_2 \text{cis}(\theta_1 + \theta_2) \tag{8}$$

Thus the rule for multiplying two complex numbers is: take the product of the magnitudes and the sum of the arguments.

A similar rule for division can be found as follows.

$$\frac{z_1}{z_2} = \frac{r_1 (\cos\theta_1 + i\sin\theta_1)}{r_2 (\cos\theta_2 + i\sin\theta_2)} \tag{9}$$
$$= \frac{r_1}{r_2} \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \tag{10}$$
$$= \frac{r_1}{r_2} (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)) \tag{11}$$
$$= \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \tag{12}$$
$$= \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2) \tag{13}$$

That is, the quotient is found by taking the quotient of the magnitudes and the difference of the arguments.

We can give a few examples of complex numbers in polar form.

Example 1. $-\frac{1}{2} = \frac{1}{2}\text{cis}\pi$.

Example 2. $-3 + 3i$.

The argument is in the second quadrant, so

$$\text{Arg} z = \arctan \frac{3}{-3} = \frac{3\pi}{4} \tag{14}$$

The magnitude is

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} \tag{15}$$

so we have

$$-3 + 3i = \sqrt{18}\text{cis}\left(\frac{3\pi}{4}\right) \tag{16}$$

Example 3. $-\pi i = \pi \text{cis}\left(-\frac{\pi}{2}\right)$.

Example 4. $-2\sqrt{3}-2i$. The argument is in the third quadrant.

$$\operatorname{Arg} z = \arctan\frac{-2}{-2\sqrt{3}} = -\frac{5\pi}{6} \tag{17}$$

$$r = \sqrt{\left(-2\sqrt{3}\right)^2 + (-2)^2} = 4 \tag{18}$$

$$z = 4\operatorname{cis}\left(-\frac{5\pi}{6}\right) \tag{19}$$

Example 5. $(1-i)\left(-\sqrt{3}+i\right)$. We can work out the arguments and magnitudes of the two factors separately and then add them. The argument of the first factor is in the fourth quadrant, and of the second factor is in the second quadrant. We have

$$1-i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right) \tag{20}$$

$$-\sqrt{3}+i = 2\operatorname{cis}\left(\frac{5\pi}{6}\right) \tag{21}$$

Thus

$$(1-i)\left(-\sqrt{3}+i\right) = 2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right) \tag{22}$$

Example 6. $\left(\sqrt{3}-i\right)^2$. We have

$$\sqrt{3}-i = 2\operatorname{cis}\left(-\frac{\pi}{6}\right) \tag{23}$$

$$\left(\sqrt{3}-i\right)^2 = 4\operatorname{cis}\left(-\frac{\pi}{3}\right) \tag{24}$$

Example 7. $\frac{-1+\sqrt{3}i}{2+2i}$. Find each factor separately and take the difference. The numerator is in the second quadrant and the denominator is in the first quadrant, so

$$-1+\sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3} \tag{25}$$

$$2+2i = 2\sqrt{2}\operatorname{cis}\frac{\pi}{4} \tag{26}$$

$$\frac{-1+\sqrt{3}i}{2+2i} = \frac{\sqrt{2}}{2}\operatorname{cis}\frac{5\pi}{12} \tag{27}$$

Example 8. $\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i}$. The numerator is in the third quadrant and the denominator is in the first quadrant, so

$$-\sqrt{7}(1+i) = \sqrt{14}\text{cis}\left(-\frac{3\pi}{4}\right) \tag{28}$$

$$\sqrt{3}+i = 2\text{cis}\frac{\pi}{6} \tag{29}$$

$$\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i} = \frac{\sqrt{14}}{2}\text{cis}\left(-\frac{11\pi}{12}\right) \tag{30}$$

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