COMPLEX NUMBERS

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The notion of a complex number arises from the question of what is $\sqrt{-1}$? Obviously no real (in the mathematical sense) number can be the square root of a negative number, so the easiest way to answer the question is just to invent a new type of number. The symbol i is usually used to indicate $i=\sqrt{-1}$ (although in some areas of physics, notably electronics, the symbol j is often used, since i is reserved for electric current), and is taken as the unit of *imaginary* numbers.

Once i has been introduced, a *complex number* can be defined as a number that is the sum of a real and an imaginary part. So we can write

$$z=a+bi \tag{1}$$

where z is a complex number, and a and b are real numbers.

The usual rules of arithmetic apply to complex numbers, so they can be added, subtracted, multiplied and divided. In addition and subtraction, the real and imaginary parts are treated separately, so

$$\begin{aligned} w &= a+bi && (2)\\ z &= c+di && (3)\\ w+z &= (a+c)+(b+d)i && (4)\\ w-z &= (a-c)+(b-d)i && (5) \end{aligned}$$

Multiplication can be done by expanding the product and collecting terms, using the usual rules of algebra:

$$\begin{aligned} wz &= (a+bi)(c+di) && (6)\\ &= ac+adi+bci+bdi^2 && (7)\\ &= (ac-bd)+(ad+bc)i && (8) \end{aligned}$$

where we used the fact that $i^2=-1$ to get the last line.

Division can be done by converting the denominator into a real number:

$$
\begin{aligned}
\frac{w}{z} &= \frac{a+bi}{c+di} && (9)\\
&= \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} && (10)\\
&= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} && (11)\\
&= \frac{ac+bd}{c^2+d^2}+\frac{(bc-ad)}{c^2+d^2}i && (12)
\end{aligned}
$$

A remarkable property of complex numbers is that any complex number can be expressed in the form

$$z = A(\cos\theta + i\sin\theta) \tag{13}$$

where A is a non-negative real number and θ is an angle in the range $[0, 2\pi]$. By comparing with the original form for z we see that

$$
\begin{aligned}
z &= a+bi && (14)\\
&= A(\cos\theta + i\sin\theta) && (15)\\
a &= A\cos\theta && (16)\\
b &= A\sin\theta && (17)
\end{aligned}
$$

The non-negative quantity A is called the modulus of z and is sometimes written as $|z|$.

The *complex conjugate* of a complex number z, written as z^* or $\bar{z}$, is obtained by reversing the sign of z's imaginary part:

$$
\begin{aligned}
z^* &= a-bi && (18)\\
&= A(\cos\theta - i\sin\theta) && (19)
\end{aligned}
$$

The product of a complex number and its complex conjugate is always a non-negative real number, and is called the *square modulus*:

$$
\begin{aligned}
z^*z &= (a-bi)(a+bi) && (20)\\
&= a^2+b^2+(ab-ba)i && (21)\\
&= a^2+b^2 && (22)\\
&= A^2 && (23)
\end{aligned}
$$

An even more remarkable identity is known as Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{24}$$

This highly non-intuitive formula can be proved by defining a function $f(\theta) = \cos\theta + i\sin\theta$ and taking its derivative:

$$\frac{df}{d\theta} = -\sin\theta + i\cos\theta \tag{25}$$

$$\frac{df}{d\theta} = if(\theta) \tag{26}$$

The equation on the last line can be integrated

$$\int \frac{df}{f} = \int i\,d\theta \tag{27}$$

$$\ln f(\theta) = i\theta + \ln C \tag{28}$$

$$f(\theta) = Ce^{i\theta} \tag{29}$$

In order to make this agree with the original definition of $f(\theta) = \cos\theta + i\sin\theta$, we compare the values at $\theta = 0$ at which we must have $f(0) = 1 = C$, so this proves Euler's formula. There are alternative proofs involving power series.

Note that the square modulus of $e^{i\theta}$is always 1:

$$|e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta \tag{30}$$

$$= 1 \tag{31}$$

A special case of Euler's formula is

$$e^{i\pi} = -1 \tag{32}$$

which is notable because it combines four of the fundamental quantities in mathematics into one simple formula.

We can therefore write any complex number as

$$z = Ae^{i\theta} \tag{33}$$

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