COMPLEX POWERS

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The definition of the power of a complex number is motivated by the complex exponential and its relation to the complex logarithm. The definition is

$$z^{\alpha} \equiv e^{\alpha \log z} \tag{1}$$

Note that the logarithm in the exponent is the general logarithm, not the principal value (Log z). This means that, in general, the power of a complex number can have more than one (possibly an infinite number) value, with one value for each branch of the logarithm.

Example 1. Find i^i. We have

$$i^i = e^{i \log i} \tag{2}$$

The logarithm has the values

$$\log i = \text{Log } |i| + i \arg i \tag{3}$$
$$= 0 + i\left(\frac{\pi}{2} + 2k\pi\right) \tag{4}$$

where $k = 0, \pm 1, \pm 2, \ldots$, so

$$i^i = e^{-(\pi/2 + 2k\pi)} \tag{5}$$

This gives the rather surprising (well it was to me anyway) conclusion that all values of i^i are real numbers, and that there are an infinite number of them, located in the interval $(0, \infty)$. The principal value is with $k = 0$:

$$i^i = e^{-\pi/2} \tag{6}$$

Example 2. Find $(-1)^{2/3}$. We have

$$(-1)^{2/3} = e^{2(\log(-1))/3} \tag{7}$$

The exponent is

$$\frac{2}{3}\log(-1) = \frac{2}{3}\left[\mathrm{Log}\,|-1| + i\arg(-1)\right] \quad (8)$$

$$= \frac{2i}{3}(\pi + 2k\pi) \quad (9)$$

$$= \frac{2\pi i}{3}(2k+1) \quad (10)$$

The principal value, with $k=0$, is

$$e^{2i\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad (11)$$

The other values will cycle through 3 values as k increases from 0, since

$$\frac{2\pi i}{3}(2k+1) = \frac{2\pi i}{3}, 2\pi i, \frac{10\pi i}{3} = \frac{4\pi i}{3} + 2\pi i, \frac{14\pi i}{3} = \frac{2\pi i}{3} + 4\pi i, \frac{18\pi i}{3} = 6\pi i, \ldots \quad (12)$$

Thus the possible values are

$$(-1)^{2/3} = e^{2\pi i/3}, e^{2\pi i}, e^{4\pi i/3} \quad (13)$$

$$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), 1, \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \quad (14)$$

We get the same 3 values if $k = -1, -2, \ldots$

Example 3. Find $2^{\pi i}$. We have

$$2^{\pi i} = e^{\pi i \log 2} \quad (15)$$

The exponent is

$$\pi i \log 2 = \pi i\left(\mathrm{Log}\ 2 + 2k\pi i\right) \quad (16)$$

$$= \pi i \mathrm{Log}\ 2 - 2k\pi^2 \quad (17)$$

so

$$2^{\pi i} = e^{-2k\pi^2} e^{\pi i \mathrm{Log}\ 2} \quad (18)$$

$$= e^{-2k\pi^2}\left(\cos(\pi \mathrm{Log}\ 2) + i\sin(\pi \mathrm{Log}\ 2)\right) \quad (19)$$

The magnitude thus ranges over values in the interval $(0,\infty)$, but the values are all complex numbers. The principal value is with $k=0$:

$$2^{\pi i}=\cos(\pi\text{Log }2)+i\sin(\pi\text{Log }2) \tag{20}$$

Example 4. Find $(1+i)^{1-i}$. We have

$$(1+i)^{1-i}=e^{(1-i)\log(1+i)} \tag{21}$$

The exponent is

$$(1-i)\log(1+i)=(1-i)(\text{Log}|1+i|+i\arg(1+i)) \tag{22}$$
$$=(1-i)\left(\text{Log }\sqrt{2}+\left(\frac{\pi}{4}+2k\pi\right)i\right) \tag{23}$$
$$=\left[\text{Log }\sqrt{2}+\frac{\pi}{4}+2k\pi\right]+i\left(\frac{\pi}{4}+2k\pi-\text{Log }\sqrt{2}\right) \tag{24}$$

so

$$(1+i)^{1-i}=e^{\text{Log }\sqrt{2}+\left(\frac{\pi}{4}+2k\pi\right)}\left[\cos\left(\frac{\pi}{4}+2k\pi-\text{Log }\sqrt{2}\right)+i\sin\left(\frac{\pi}{4}+2k\pi-\text{Log }\sqrt{2}\right)\right] \tag{25}$$
$$=\sqrt{2}e^{\frac{\pi}{4}+2k\pi}\left[\cos\left(\frac{\pi}{4}-\text{Log }\sqrt{2}\right)+i\sin\left(\frac{\pi}{4}-\text{Log }\sqrt{2}\right)\right] \tag{26}$$

Using the identities

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right)&=\sin\theta\\ \sin\left(\frac{\pi}{2}-\theta\right)&=\cos\theta\end{aligned} \tag{27}$$

we have the equivalent form

$$(1+i)^{1-i}=\sqrt{2}e^{\frac{\pi}{4}+2k\pi}\left[\sin\left(\frac{\pi}{4}+\text{Log }\sqrt{2}\right)+i\cos\left(\frac{\pi}{4}+\text{Log }\sqrt{2}\right)\right] \tag{28}$$

Again, we have an infinite number of values as the $2k\pi$ component in the exponent varies over all integers. The principal value is with $k=0$:

$$(1+i)^{1-i}=\sqrt{2}e^{\frac{\pi}{4}}\left[\cos\left(\frac{\pi}{4}-\text{Log }\sqrt{2}\right)+i\sin\left(\frac{\pi}{4}-\text{Log }\sqrt{2}\right)\right] \tag{29}$$

Example 5. Find $(1+i)^3$. We have

$$(1+i)^3=e^{3\log(1+i)} \tag{30}$$

The exponent is

$$3\log(1+i) = 3\left[\text{Log }|1+i| + i\arg(1+i)\right] \tag{31}$$
$$= 3\left[\text{Log }\sqrt{2} + \left(\frac{\pi}{4} + 2k\pi\right)i\right] \tag{32}$$

so

$$(1+i)^3 = e^{3\text{Log }\sqrt{2}}e^{\left(\frac{3\pi}{4}+6k\pi\right)i} \tag{33}$$

In this case, varying k will always just add a multiple of $6\pi i$ to the exponent, so it will make no difference. Thus there is only one solution:

$$(1+i)^3 = e^{3\text{Log }\sqrt{2}}e^{3\pi i/4} \tag{34}$$
$$= 2^{3/2}\left(-2^{-1/2} + 2^{-1/2}i\right) \tag{35}$$
$$= -2 + 2i \tag{36}$$

It would probably be easier to just multiply out $(1+i)^3$ using the binomial theorem, but using logarithms shows that the answer is unique.

Example 6. Is 1 raised to any power α (real or complex) always equal to 1? We have

$$1^\alpha = e^{\alpha\log 1} \tag{37}$$

The exponent is

$$\alpha\log 1 = \alpha\left(\text{Log }1 + i\arg 1\right) \tag{38}$$
$$= i\alpha 2k\pi \tag{39}$$

Thus if α is a positive or negative integer, the result of 1^α is always 1. However, for any other values, the result is not always 1. For example, the nth root of 1 gives n different values. If α is imaginary, then we have

$$1^\alpha = e^{-2k\pi|\alpha|} \tag{40}$$

so in this case 1^α has an infinite number of real values.