CONSTANT COMPLEX FUNCTIONS

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Using the Cauchy-Riemann equations (CR) we can show that certain types of complex functions must be constant over a domain D. In what follows, we write a complex function as

$$f(z) = u(x,y) + iv(x,y) \tag{1}$$

and the Cauchy-Riemann equations are

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{2}$$

To streamline the notation we'll use the shorthand

$$\begin{aligned} u_x &\equiv \frac{\partial u}{\partial x} \\ u_y &\equiv \frac{\partial u}{\partial y} \end{aligned} \tag{3}$$

and similarly for the derivatives of v.

Theorem 1. *For an analytic function $f(z)$, if either $\Re f(z)$ or $\Im f(z)$ is constant over a domain D, then $f(z)$ must be constant over D.*

Proof. If $\Re f(z) = u(x,y)$ is constant, then $u_x = u_y = 0$ so from 2 we must also have $v_x = v_y = 0$, so both $u(x,y)$ and $v(x,y)$ are constant functions. Thus $f(z)$ is also constant. □

Theorem 2. *If both $f(z)$ and $\overline{f(z)}$ are analytic in a domain D, then $f(z)$ must be constant in D.*

Proof. We have

$$f(z) = u(x,y) + iv(x,y) \tag{4}$$

$$\overline{f(z)} = u(x,y) - iv(x,y) \tag{5}$$

Applying CR to $f(z)$ gives us the conditions 2. Applying these conditions to $\overline{f(z)}$ we just change the sign on v, so we get

$$\begin{aligned} u_x &= -v_y \\ u_y &= v_x \end{aligned} \tag{6}$$

Combining the two conditions gives us

$$u_x = v_y = -v_y \tag{7}$$
$$u_y = -v_x = v_x \tag{8}$$

Thus we must have $v_x = v_y = 0$, from which 6 gives us $u_x = u_y = 0$, so both u and v are constants. ☐

Theorem 3. *If $f(z)$ is analytic in a domain D and $|f(z)|$ is constant in D, then $f(z)$ itself is also constant in D.*

Proof. To see this, consider the derivatives of $|f(z)|^2$. Since $|f(z)|$ is constant, its square is also constant, so we have $|f|^2 = u^2 + v^2$ and

$$\begin{aligned} \frac{\partial |f|^2}{\partial x} &= 0 = 2uu_x + 2vv_x \\ \frac{\partial |f|^2}{\partial y} &= 0 = 2uu_y + 2vv_y \end{aligned} \tag{9}$$

Applying CR to these equations to eliminate the derivatives of v gives, after cancelling the 2

$$uu_x - vu_y = 0 \tag{10}$$
$$uu_y + vu_x = 0 \tag{11}$$

Solving for u_x gives

$$u_x = \frac{vu_y}{u} = -\frac{uu_y}{v} \tag{12}$$

Multiplying through by uv and rearranging we have

$$\left(v^2 + u^2\right) u_y = 0 \tag{13}$$

Thus either $v^2 + u^2 = 0$, in which case $u = v = 0$ and $f(z) = 0$ everywhere (and is therefore constant)[1], or $u_y = 0$. If the latter, then from 12, $u_x = 0$

[1] OK, technically this equation isn't valid if $u = v = 0$ since we divided by u and v in 12. However, we can see that $u = v = 0$ is a solution of 9 so we're safe.

so u is constant. From 9 this implies that $v_x = v_y = 0$ so v is also constant, meaning that f is constant. $\square$