CONSTANT FUNCTION ON A DOMAIN

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A domain D is a set in the complex plane that is open and connected. This means that any point in D has a neighbourhood that is also in D and there is a polygonal path between any two points in D. We can prove the following theorem about a function defined over the domain D.

Theorem 1. *Let $U(x,y)$ be a function of two variables defined within a domain D. Then if*

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial y}=0 \tag{1}$$

everywhere in D, U is constant within D.

Proof. First, we can express a line segment in parametric form. From linear algebra, a line L can be written as a vector sum $\boldsymbol{p}+t\boldsymbol{v}$, where the vector $\boldsymbol{p}$ is a fixed point on L, $\boldsymbol{v}$ is a constant vector parallel to L and t is a parameter which, for an infinite line, extends from $-\infty$ to ∞. In terms of components, we can write

$$\boldsymbol{p}=(b,d) \tag{2}$$

$$\boldsymbol{v}=(a,c) \tag{3}$$

Note that a,b,c,d are all constants and t is the only variable. In rectangular coordinates, the parameteric form then becomes

$$\begin{aligned} x&=b+at\\ y&=d+ct \end{aligned} \tag{4}$$

For a finite line segment, we can choose $\boldsymbol{p}$ to point to one end, say $\boldsymbol{p}_1$, of the segment and $\boldsymbol{p}+\boldsymbol{v}$ to point to the other end. Points between the endpoints are then found by varying t from 0 to 1.

We now write U in terms of the parametric form of the line segment as

$$U(x,y)=U(b+at,d+ct) \tag{5}$$

To show that U is constant in D, we find the total derivative $\frac{dU}{dt}$ using the chain rule for derivatives. We have

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} \tag{6}$$

$$= a\frac{\partial U}{\partial x} + c\frac{\partial U}{\partial y} \tag{7}$$

$$= 0 \tag{8}$$

where the last line follows from the theorem's assumption 1. Thus U is constant along any given line segment. The constancy of U over all of D follows from the fact that D is a domain and is therefore connected. That is, we can reach any point by means of a connected polygonal path. Suppose this path is composed of line segments $(s_1, s_2, \ldots, s_n)$. We know that U is constant along s_1 with value u_1, say. Since s_2 is connected to s_1 and U is constant along s_2 and has value u_1 at the point where s_1 joins s_2, it must have the same value u_1 along s_2. The same argument applies for all line segments, thus U has the value u_1 over all of D. □