CONTINUOUSLY DEFORMABLE LOOPS

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The theory of complex integration involves the specification of contours and, more specifically, closed contours or loops. Many theorems involve the integration of functions around such loops, so it's important to understand some of the basic terminology. A more complete discussion is given in Saff and Snider's book, so here I'll summarize the main ideas.

First, we need to specify the domain D in which we're interested. In general, a domain is some region of the complex plane over which a function $f(z)$ is well-defined. A loop within a domain is a closed contour that lies entirely within D. A loop Γ_0 in D can be continuously deformed into another loop $\Gamma_1$if, by stretching and/or translating Γ_0 it can be deformed into Γ_1 with every intermediate stage in the transformation occurring entirely within D. You can think of a deformation by imagining Γ_0 to be an elastic band. If it's possible to expand or shrink the elastic band and/or slide it across the complex plane, all the time without leaving D, and thus transform it into Γ_1, then Γ_0 is continuously deformable. We can reverse the process and transform Γ_1 into Γ_0 by performing the deformation in reverse order.

Example 1. Consider D to be the annulus $1<|z|<10$. The loop Γ_0 defined by $|z|=5$ is the circle of radius 5 centred at the origin. We can deform this circle to the loop Γ_1 given by $|z|=6$ by expanding Γ_0 until its radius is 6. However, if we try to deform Γ_0 to the loop Γ_2 given by $|z-7|=4$, we cannot do this, because part of Γ_2 lies outside D.

Example 2. A single point P is an extreme example of a loop. Thus if we consider the domain D to be the entire complex plane, then any loop within D can be deformed by shrinking and then moving it to coincide with any point P in the plane. Within the annulus of Example 1, however, the loop Γ_0 cannot be deformed to a single point without leaving D, since we would need to shrink it down below the lower limit $|z|=1$.

The notion of domains that allow deformations to a point is formalized in the definition:

Definition 1. A domain D possessing the property that every loop in D can be continuously deformed to a single point is called a *simply connected domain*.

In simple language, a simply connected domain is a region of the complex plane that doesn't have any holes in it. If a domain has a hole, then if we draw a loop around the hole, we can't deform the loop to a single point since we would need to enter the hole in order to do so.

A continuous deformation of Γ_0 to Γ_1 can be thought of as a continuous sequence of intermediate loops that starts at Γ_0 and morphs into Γ_1. We can formalize this notion with the following definition.

Definition 2. A loop Γ_0 is said to be *continuously deformable* to the loop Γ_1 in a domain D if there exists a function $z(s,t)$ continuous in the ranges $0 \le s \le 1$ and $0 \le t \le 1$ that satisfies the following conditions

(1) For each fixed value of s in $[0,1]$, the function $z(s,t)$ parametrizes a loop lying entirely within D. That is, if we fix $s = s_0$, say, then the function $z(s_0,t)$ is a parametric representation of a loop as t varies from 0 to 1.

(2) The function $z(0,t)$ parametrizes Γ_0.

(3) The function $z(1,t)$ parametrizes Γ_1.

Thus each value of s between 0 and 1 gives a parametric equation describing a loop that is intermediate in form between Γ_0 and Γ_1.

Example 3. Suppose we have the two loops in the complex plane given by:

$$\Gamma_0: \quad \frac{x^2}{4}+\frac{y^2}{9}=1 \quad \text{(ellipse)} \tag{1}$$

$$\Gamma_1: \quad x^2+y^2=1 \quad \text{(circle)} \tag{2}$$

Since we're considering D to be the entire complex plane, we can envision a continuous deformation of the ellipse (centred at the origin) to the circle (also centred at the origin).

We start with a parametric form for the ellipse, which can be written as

$$z(t) = 2\cos 2\pi t + 3i\sin 2\pi t \qquad 0 \le t \le 1 \tag{3}$$

That is

$$\begin{aligned} x(t) &= 2\cos 2\pi t \\ y(t) &= 3\sin 2\pi t \end{aligned} \tag{4}$$

We would like to transform this to

$$\begin{aligned} x(t) &= \cos 2\pi t \\ y(t) &= \sin 2\pi t \end{aligned} \tag{5}$$

We need a couple of functions of s to achieve this. We would like to find functions $f(s)$ and $g(s)$ such that

$$x(s,t) = 2f(s)\cos 2\pi t \tag{6}$$
$$y(s,t) = 3g(s)\sin 2\pi t \tag{7}$$

with the conditions

$$f(0) = 1 \tag{8}$$
$$f(1) = \frac{1}{2} \tag{9}$$
$$g(0) = 1 \tag{10}$$
$$g(1) = \frac{1}{3} \tag{11}$$

If we take these functions to be linear, then we have

$$f(s) = as + b \tag{12}$$
$$g(s) = cs + d \tag{13}$$

(other forms for f and g are possible, but linear seems the simplest).

Inserting the boundary conditions, we have

$$f(0) = b = 1 \tag{14}$$
$$f(1) = a + 1 = \frac{1}{2} \tag{15}$$
$$g(0) = d = 1 \tag{16}$$
$$g(1) = c + 1 = \frac{1}{3} \tag{17}$$

Therefore

$$a = -\frac{1}{2} \tag{18}$$

$$b = 1 \tag{19}$$

$$c = -\frac{2}{3} \tag{20}$$

$$d = 1 \tag{21}$$

and we have

$$z(s,t) = 2\left(1-\frac{s}{2}\right)\cos 2\pi t + 3i\left(1-\frac{2s}{3}\right)\sin 2\pi t \tag{22}$$

As s varies between 0 and 1, we get a series of ellipses that shrink from the initial ellipse 1 to the circle 2. Each intermediate ellipse has the equation

$$\frac{x^2}{4\left(1-\frac{s}{2}\right)^2} + \frac{y^2}{9\left(1-\frac{2s}{3}\right)^2} = 1 \tag{23}$$

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