CONTOUR INTEGRALS

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Because complex numbers cover a plane rather than the real line needed for real numbers, in general an integral of a complex function between two points in the complex plane needs to follow some path between the two points. It might seem that the result of such an integral would depend on the path taken, but in fact it does not. However, we'll get to this in a future post. Here, we'll consider the definition of a integral over some specific path or contour between two points in the complex plane.

The general idea is similar to that used in defining an integral in real calculus[1]. Suppose we have a complex function $f(z)$ that is continuous over some contour Γ. We can partition the contour into a number of smooth curves, $\gamma_1, \gamma_2, \ldots, \gamma_n$. The endpoints of curve γ_k are then defined to be z_{k-1} and z_k.

If we make the length of each curve small enough, then this length is approximately $z_k - z_{k-1}$. If c_k is the midpoint of γ_k, we can define the *Riemann sum* for the function $f(z)$ for the partition above (which we refer to as $\mathscr{P}_n$) as

$$S(\mathscr{P}_n) \equiv \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) \tag{1}$$

$$= \sum_{k=1}^{n} f(c_k)\Delta z_k \tag{2}$$

If we have an admissible parametrization of the contour Γ then each smooth curve γ_k within Γ can be written as a function $z(t_k)$ of some parameter t_k. The length Δz_k is then approximately

$$\Delta z_k \approx z'(t_k)\Delta t_k \tag{3}$$

and the Riemann sum 2 becomes

[1]The description given here is an outline of the general idea. A more complete derivation is given in Saff and Snider's book.

$$S(\mathscr{P}_n)=\sum_{k=1}^{n}f(z(t_k))\,z'(t_k)\,\Delta t_k \tag{4}$$

In the limit as all the $\Delta t_k\to 0$ and $n\to\infty$, the Riemann sum becomes an integral.

$$\lim_{n\to\infty}S(\mathscr{P}_n)=\int_a^b f(z(t))\,z'(t)\,dt \tag{5}$$

where $a\le t\le b$.

The complete integral over the entire contour Γ is then the sum of the integrals over the smooth curves γ_j that make up Γ.

$$\int_\Gamma f(z)\,dz=\sum_j\int_{\gamma_j}f(z)\,dz \tag{6}$$

An important point in 5 is that the integral is the same for *any* admissible parametrization of the curve. This follows from the formula for rescaling the parameter t given earlier

$$z(t)=z\left(\frac{b-a}{d-c}t+\frac{ad-bc}{d-c}\right) \tag{7}$$

where $a\le t\le b$ in one parametrization and $c\le t\le d$ in another. See the earlier post for details.

Example 1. Compute the contour integral $\int_\Gamma\overline{z}dz$ for the following contours.

(a) The circle $|z|=2$ traversed once counterclockwise. We need a suitable parametrization of Γ. For a circle of radius 2 centred at the origin we have

$$z(t)=2e^{it} \tag{8}$$

with $0\le t\le 2\pi$ for a counterclockwise traversal. We have

$$f(z)=\overline{z}(t)=2e^{-it} \tag{9}$$
$$z'(t)=2ie^{it} \tag{10}$$

Therefore

$$\int_{\Gamma} f(z)\,dz = \int_0^{2\pi} \left(2e^{-it}\right)\left(2ie^{it}\right) dt \tag{11}$$
$$= \int_0^{2\pi} 4i\;dt \tag{12}$$
$$= 8\pi i \tag{13}$$

(b) The circle $|z| = 2$ traversed once clockwise. This is the same as part (a) except the limits on the integral are reversed, so the answer is $-8\pi i$. To verify this explicitly, for a clockwise traversal

$$z(t) = 2e^{-it} \tag{14}$$
$$z'(t) = -2ie^{-it} \tag{15}$$

so

$$f(z) = \overline{z}(t) = 2e^{it} \tag{16}$$

Therefore

$$\int_{\Gamma} f(z)\,dz = \int_0^{2\pi} \left(2e^{it}\right)\left(-2ie^{-it}\right) dt \tag{17}$$
$$= -\int_0^{2\pi} 4i\;dt \tag{18}$$
$$= -8\pi i \tag{19}$$

(c) The circle $|z| = 2$ traversed three times clockwise. We can regard this contour as three instances of the contour in part (b), so the answer is $-24\pi i$.

Example 2. Evaluate $\int_{\Gamma}(x - 2xyi)\,dz$ over the contour $\Gamma : z = t + it^2$ for $0 \le t \le 1$. We have

$$z(t) = t + it^2 \tag{20}$$
$$z'(t) = 1 + 2it \tag{21}$$

$$f(z) = x - 2xyi \tag{22}$$

Since the function to be integrated is given in terms of its real and imaginary parts, we'll need to integrate these separately. We have

$$\begin{aligned} x &= t \\ y &= t^2 \end{aligned} \tag{23}$$

From 5 we have

$$\int_{\Gamma} f(z)\,dz = \int_0^1 \left(t-2t^3 i\right)(1+2ti)\,dt \tag{24}$$

$$= \int_0^1 \left(t+4t^4\right)dt + \int_0^1 \left(2t^2-2t^3\right) i\; dt \tag{25}$$

$$= \left[\frac{t^2}{2}+\frac{4}{5}t^5+\left(\frac{2}{3}t^3-\frac{1}{2}t^4\right)i\right]_0^1 \tag{26}$$

$$= \frac{1}{2}+\frac{4}{5}+\left(\frac{2}{3}-\frac{1}{2}\right)i \tag{27}$$

$$= \frac{13}{10}+\frac{1}{6}i \tag{28}$$

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