

CONTOUR INTEGRATION AROUND A BRANCH CUT

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We can make use of contour integration with branch cuts to evaluate some real integrals. To do this we use the complex logarithm $\log z$ which is multivalued in that

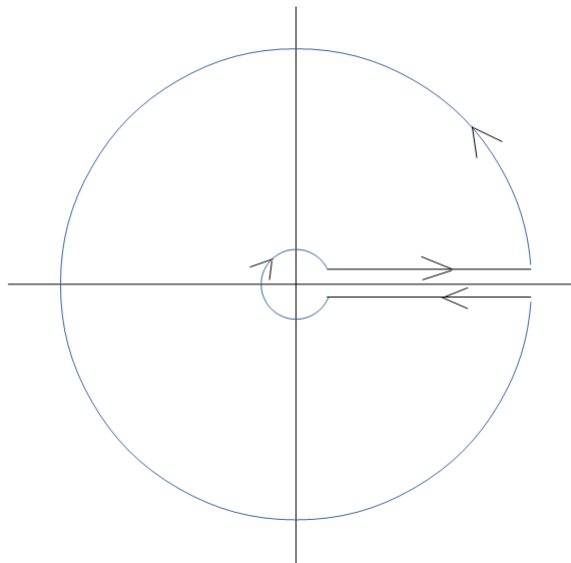
$$(1) \quad \log z = \log (re^{i\theta}) = \log r + i\theta$$

However, if we rotate θ by 2π around $z = 0$, the value of z returns to its original value, but the logarithm does not:

$$(2) \quad \log (re^{i(\theta+2\pi)}) = \log r + i\theta + 2\pi i$$

$\log z$ has branch points at $z = 0$ and $z = \infty$, so to restrict $\log z$ to a single valued function, we can use a branch cut on the positive real axis.

If we want to do a contour integral of any function containing $\log z$, the contour must avoid crossing the branch cut. A commonly used contour is the *keyhole* contour shown:



To see how this contour can be used, suppose we want to find a real integral of some function $f(x)$ along the positive real axis, that is we want to find

$$(3) \quad I = \int_0^{\infty} f(x) dx$$

If $f(x)$ is well-behaved on the real axis, we can generalize it to a complex function $f(z)$, and then consider the integral of the function $g(z) = f(z)\log z$ around the contour above. If we can show that the integral of $g(z)$ over the outer and inner circular portions of the contour goes to zero as the radii of these circles tends to infinity (for the outer circle) and zero (for the inner circle), then we're left with the integrals over the two straight line portions on either side of the positive real axis. Just above the axis, $\theta \rightarrow 0$ so $z = xe^{i0}$. Just below the axis, however, $\theta \rightarrow 2\pi$ and $z = xe^{2\pi i}$. Therefore, the integrals on these two lines are

$$(4) \quad I_{upper} = \int_0^{\infty} f(x) \log x dx$$

$$(5) \quad I_{lower} = -\int_0^{\infty} f(x) (\log x + 2\pi i) dx$$

[The minus sign in I_{lower} is because the direction of integration is from right to left.] Adding these together and applying the residue theorem, we get

$$(6) \quad \int_{contour} f(z) \log z dz = I_{upper} + I_{lower}$$

$$(7) \quad = -2\pi i \int_0^{\infty} f(x) dx$$

$$(8) \quad = 2\pi i \sum_i \text{Res}(f(z) \log z, a_k)$$

where a_k is the k th pole of $f(z)$. It's important to remember that this result works only if $f(z)$ is well behaved on the real axis, and the integrals over the two circular sections of the contour tend to zero in the limit. However, these conditions are satisfied for a large collection of functions.

Example. Find $I = \int_0^{\infty} \frac{dx}{1+x^4}$. Generalizing to a complex function, we have

$$(9) \quad f(z) = \frac{1}{1+z^4}$$

which has four poles at

$$\begin{aligned}
 (10) \quad a_k &= (-1)^{1/4} \\
 (11) \quad &= \left(e^{i\pi+2k\pi} \right)^{1/4} \\
 (12) \quad &= e^{i\pi/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}
 \end{aligned}$$

To find the integral of $g(z) = \log z / (1 + z^4)$ around the contour, we need the residues of $g(z)$ at the four poles. The residues are

$$\begin{aligned}
 (13) \quad \text{Res}(g(z), a_k) &= \lim_{z \rightarrow a_k} (z - a_k) g(z) \\
 (14) \quad &= \lim_{z \rightarrow a_k} (z - a_k) f(z) \log z
 \end{aligned}$$

Since $\log z$ is well behaved at all four poles, we can consider

$$(15) \quad \lim_{z \rightarrow a_k} (z - a_k) f(z) = \lim_{z \rightarrow a_k} \frac{(z - a_k)}{1 + z^4}$$

These limits can be found using l'Hôpital's rule, which says that if $\lim_{z \rightarrow a_k} f(z) = \lim_{z \rightarrow a_k} h(z) = 0$, then

$$(16) \quad \lim_{z \rightarrow a_k} \frac{f(z)}{h(z)} = \lim_{z \rightarrow a_k} \frac{f'(z)}{h'(z)}$$

Applying this gives us

$$(17) \quad \lim_{z \rightarrow a_k} \frac{(z - a_k)}{1 + z^4} = \frac{1}{4a_k^3}$$

$$(18) \quad = \frac{a_k}{4a_k^4}$$

$$(19) \quad = -\frac{a_k}{4}$$

since $a_k^4 = -1$ at all four poles. Therefore, the residues are

$$(20) \quad \text{Res}(f(z) \log z, a_k) = -\frac{1}{4} a_k \log a_k$$

$$(21) \quad = -\frac{1}{4} e^{i\pi/4} e^{ik\pi/2} \left[\frac{\pi i}{4} (1 + 2k) \right]$$

$$(22) \quad = -\frac{\pi i}{16} e^{i\pi/4} e^{ik\pi/2} (1 + 2k)$$

The four residues are

| k | $\text{Res}(a_k)$ |
|-----|----------------------------------------------|
| 0 | $-\frac{\pi i}{16} \frac{(1+i)}{\sqrt{2}}$ |
| 1 | $-\frac{3\pi i}{16} \frac{(-1+i)}{\sqrt{2}}$ |
| 2 | $-\frac{5\pi i}{16} \frac{(-1-i)}{\sqrt{2}}$ |
| 3 | $-\frac{7\pi i}{16} \frac{(1-i)}{\sqrt{2}}$ |

Adding them up we get

$$(23) \quad I_{\text{contour}} = 2\pi i \sum_k \text{Res}(a_k)$$

$$(24) \quad = -2\pi i \frac{\pi i}{16\sqrt{2}} [(1-3-5+7) + i(1+3-5-7)]$$

$$(25) \quad = \frac{\pi^2 i}{\sqrt{2}}$$

To get the desired integral, we still need to show that integral of $g(z)$ over the circular portions of the contour go to zero. Over the outer circle, we can set $z = Re^{i\theta}$ and let $R \rightarrow \infty$. We have

$$(26) \quad \left| g(Re^{i\theta}) \right| = \frac{|\log Re^{i\theta}|}{|1 + R^4 e^{4i\theta}|}$$

Integrating over the outer circle gives

$$(27) \quad \int_R \frac{|\log Re^{i\theta}|}{|1 + R^4 e^{4i\theta}|} dz = \int_0^{2\pi} R \frac{|\log Re^{i\theta}|}{|1 + R^4 e^{4i\theta}|} d\theta$$

$$(28) \quad < \int_0^{2\pi} R \frac{\log R + \theta}{|R^4 - 1|} d\theta$$

$$(29) \quad < 2\pi R \frac{\log R + 2\pi}{|R^4 - 1|} \rightarrow 0$$

In the second line, we want to maximize the integrand (and show it still goes to zero as $R \rightarrow \infty$), so we need to minimize the denominator, which occurs when $e^{4i\theta} = -1$. Thus the integral over the outer circle does indeed go to zero.

For the inner circle, we can take the radius to be ε and get the same expression:

$$(30) \quad \int_\varepsilon \frac{|\log \varepsilon e^{i\theta}|}{|1 + \varepsilon^4 e^{4i\theta}|} dz < 2\pi \varepsilon \frac{\log \varepsilon + 2\pi}{|\varepsilon^4 - 1|} \rightarrow 0$$

since $\lim_{\varepsilon \rightarrow 0} \varepsilon \log \varepsilon = 0$.

Combining everything and using 8 and 25 we get

$$(31) \quad \int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2\pi i} I_{contour} = \frac{\pi}{2\sqrt{2}}$$

which is (fortunately) real.