

## CONTOUR INTEGRATION

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I'll collect here a few useful techniques for using contour integration in the complex plane. I'm assuming that these techniques are still taught relatively late in the physics curriculum so it's not something that may be widely known by armchair physicists. (I didn't take a course on complex variable theory until my third year as an undergraduate).

Suppose we have a function  $f(z)$  defined over some region of the complex plane. If  $f(z)$  is *holomorphic*, that is, it is differentiable (in the complex-variable sense of differentiable) over the entire region, then Cauchy's theorem says that if you integrate this function around any closed contour  $C$  within that region, the result is zero. That is:

$$\oint_C f(z) dz = 0 \quad (1)$$

Now suppose that there is one point  $a$  within the region where  $f$  is not differentiable because it blows up at that point. For example, suppose we have the function

$$f(z) = \frac{1}{z-i} \quad (2)$$

Then  $f$  is holomorphic over the entire complex plane except at the point  $z = i$ .  $f(z)$  is said to have a *pole* at  $z = i$ . The pole is a *simple pole*, because the denominator is first order in  $z - i$ . A higher order pole occurs if the denominator is higher order, so that the function

$$f(z) = \frac{1}{(z-i)^n} \quad (3)$$

has an  $n$ th order pole at  $z = i$ .

The *residue* of a function at a pole is given by

$$\text{Res}(f, a) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \quad (4)$$

For a simple pole ( $n = 1$ ), this formula reduces to

$$\operatorname{Res}(f, a) = \lim_{z \rightarrow a} [(z - a) f(z)] \quad (5)$$

For example, the residue of 2 at  $z = i$  is

$$\operatorname{Res}(f, i) = \lim_{z \rightarrow i} \frac{z - i}{z - i} = 1 \quad (6)$$

Cauchy's residue theorem then gives an elegant result (I can remember thinking this result was almost magical when I first learned it). If you choose *any* closed contour  $C$  and integrate  $f(z)$  around  $C$  in a counter-clockwise direction, the result is  $2\pi i$  times the sum of the residues at the poles within the contour. That is

$$\oint_C f(z) dz = 2\pi i \sum_k \operatorname{Res}(f, a_k) \quad (7)$$

where  $a_k$  is a pole within  $C$ .

For example, if we integrate 2 around any contour  $C$  that contains  $z = i$  we get

$$\oint_C \frac{dz}{z - i} = 2\pi i \operatorname{Res}(f, i) = 2\pi i \quad (8)$$

The most common application of Cauchy's theorems in physics is in the evaluation of definite integrals, typically involving infinite limits.

**Example 1.** Consider the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \quad (9)$$

The integral can be done using elementary calculus with the result

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \quad (10)$$

We can also do this using contour integration by defining a complex function

$$f(z) = \frac{1}{1 + z^2} = \frac{1}{(z + i)(z - i)} \quad (11)$$

This function has simple poles at  $z = \pm i$ , with residues:

$$\operatorname{Res}(f, i) = \frac{1}{2i} \quad (12)$$

$$\operatorname{Res}(f, -i) = -\frac{1}{2i} \quad (13)$$

We can choose a contour as follows. For  $R > 0$ , draw a line along the real axis from  $-R$  to  $+R$ , then from  $+R$ , draw a semicircle of radius  $R$  in the upper half plane so that it meets the real axis at  $-R$ . This contour encloses only the pole at  $z = i$ , so the integral around this contour is

$$\oint_C \frac{dz}{1+z^2} = 2\pi i \left( \frac{1}{2i} \right) = \pi \quad (14)$$

On the semicircular arc portion of the contour,  $z = Re^{it}$  where  $t$  is a *real* parameter that varies from 0 to  $\pi$  as we proceed along the arc in a counterclockwise direction. Therefore

$$dz = iRe^{it}dt \quad (15)$$

$$\int_{arc} \frac{dz}{1+z^2} = iR \int_{arc} \frac{e^{it}dt}{1+R^2e^{2it}} \quad (16)$$

If we now let  $R \rightarrow \infty$ , the integral on the RHS goes to zero, and the integral along the real axis tends to the integral in 9 so we get our result

$$\lim_{R \rightarrow \infty} \oint_C \frac{dz}{1+z^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \quad (17)$$

This example is a bit like using a sledgehammer to crack a peanut, but it's nice to verify that the residue theorem works in a simple case.

**Example 2.** A much more common case is when we wish to find a real definite integral that has singularities on the real axis. For example

$$\int_0^{\infty} \frac{\sin x}{x} dx \quad (18)$$

This function has a singularity at  $x = 0$  (actually the function is well-behaved there since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , but the indefinite integral itself has no closed form solution). We can begin by converting this to a complex function

$$f(z) = \frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z} \quad (19)$$

We can use the same contour as in Example 1, except now we bypass the point  $z = 0$  with a small semicircular arc of radius  $\epsilon$ . Thus the total contour consists of 4 sections:

$$C = \begin{cases} t \in [\epsilon, R] & z = t \\ t \in [0, \pi] & z = Re^{it} \\ t \in [-R, -\epsilon] & z = t \\ t \in [\pi, 0] & z = \epsilon e^{it} \end{cases} \quad (20)$$

[Forgive me for not providing a drawing, but they're hard to do!]

This contour bypasses the difficult point so that  $f(z)$  has no poles within the contour, thus by Cauchy's theorem

$$\oint_C \frac{e^{iz}}{z} dz = 0 \quad (21)$$

To work out the integral, we split it into the four parts of the contour:

$$\oint_C \frac{e^{iz}}{z} dz = \int_{\epsilon}^R \frac{e^{it}}{t} dt + \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} dz + \int_{-R}^{-\epsilon} \frac{e^{it}}{t} dt + \int_{\pi}^0 \frac{e^{i\epsilon e^{it}}}{\epsilon e^{it}} dz \quad (22)$$

$$= \int_{\epsilon}^R \frac{e^{it}}{t} dt + \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it} dt + \int_{-R}^{-\epsilon} \frac{e^{it}}{t} dt + \int_{\pi}^0 \frac{e^{i\epsilon e^{it}}}{\epsilon e^{it}} i\epsilon e^{it} dt \quad (23)$$

The third integral can be transformed by substituting  $t = -x$ :

$$\int_{-R}^{-\epsilon} \frac{e^{it}}{t} dt = \int_R^{\epsilon} \frac{e^{-ix}}{x} dx = - \int_{\epsilon}^R \frac{e^{-ix}}{x} dx \quad (24)$$

Changing the integration variable from  $t$  to  $x$  in the first integral in 23 and adding to this result gives

$$\int_{\epsilon}^R \frac{e^{it}}{t} dt + \int_{-R}^{-\epsilon} \frac{e^{it}}{t} dt = \int_{\epsilon}^R \frac{e^{ix} - e^{-ix}}{x} dx \quad (25)$$

$$= 2i \int_{\epsilon}^R \frac{\sin x}{x} dx \quad (26)$$

Taking the limits  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$  thus give us the required integral 18. To deal with the other two integrals in 23, we'll look first at the second integral. The numerator in the integrand is

$$e^{iRe^{it}} = e^{iR \cos t} e^{-R \sin t} \quad (27)$$

Remembering that  $t$  is real and in the range  $t \in [0, \pi]$ ,  $\sin t \geq 0$  so  $-R \sin t < 0$ . As  $R \rightarrow \infty$ , the integrand thus tends to zero exponentially, so the integral tends to zero.

For the other integral, we want the limit as  $\epsilon \rightarrow 0$ , so we have

$$\lim_{\epsilon \rightarrow 0} \int_{\pi}^0 \frac{e^{i\epsilon e^{it}}}{\epsilon e^{it}} i\epsilon e^{it} dt = i \lim_{\epsilon \rightarrow 0} \int_{\pi}^0 e^{i\epsilon e^{it}} dt = -\pi i \quad (28)$$

Combining this with 26 we have

$$\lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \oint_C \frac{e^{iz}}{z} dz = 2i \int_0^{\infty} \frac{\sin x}{x} dx - \pi i = 0 \quad (29)$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \quad (30)$$

Most infinite integrals with singularities on the real axis need to be handled this way. Bypass the singularities with little semicircular arcs and work out the integral on each segment of the contour. Typically the integral over the large semicircular arc tends to zero as  $R \rightarrow \infty$ .

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