CONTOURS

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The notion of a *contour* plays an important role in integration with complex functions. Simply stated, a contour is a set of smooth curves joined end-to-end. Thus a contour is a continuous curve that may contain cusps and corners, that is, points where the curve is not differentiable. A more formal definition is:

Definition 1. A *contour* Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for every $k = 1, 2, \ldots, n-1$. We can write the contour as

$$\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n \tag{1}$$

Just as a smooth curve can be parametrized, so also can a contour. That is, it's possible to define a function $z(t)$ of a real parameter t such that as t varies between two fixed values a and b, $z(t)$ traces out the contour. We can usually create a parametrization of a contour by scaling the parameter of each smooth curve that makes up the contour.

A general formula which can be used to transform a parameter variable t is

$$z(t) = z\left(\frac{b-a}{d-c}t + \frac{ad-bc}{d-c}\right) \tag{2}$$

where the parentheses on the RHS indicate a functional dependence, not a product. Here the parameter t is to be transformed from the range $a \le t \le b$ to $c \le t \le d$.

To verify this formula, take $t = c$ on the RHS. Then we have

$$z\left(\frac{b-a}{d-c}t + \frac{ad-bc}{d-c}\right) = z\left(\frac{bc-ac+ad-bc}{d-c}\right) \tag{3}$$

$$= z(a) \tag{4}$$

Likewise, take $t = d$ on the RHS. We have

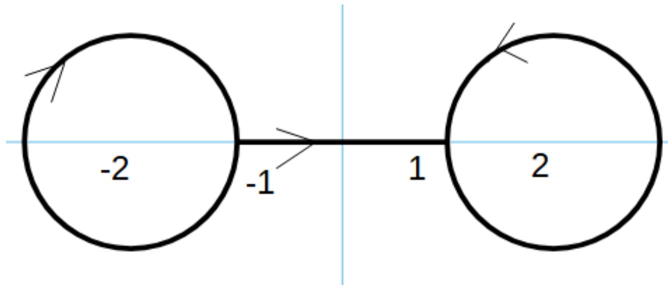

FIGURE 1. A contour consisting of 3 smooth curves.

$$z\left(\frac{b-a}{d-c}t+\frac{ad-bc}{d-c}\right)=z\left(\frac{bd-ad+ad-bc}{d-c}\right) \tag{5}$$

$$=z(b) \tag{6}$$

Thus the endpoints of the parameter match in both cases.

Example 1. Consider the contour shown in Fig. 1. We begin at the point $z=-1$ and proceed clockwise around the circle of radius 1 centred at $z=-2$ until we come back to $z=-1$. Then we follow the straight line segment to $z=1$, after which we proceed counterclockwise around the circle of radius 1 centred at $z=2$.

We start with individual parametrizations for each of the three curves. For the left circle, we have

$$z(t)=-2+e^{-it} \qquad (0\le t\le 2\pi) \tag{7}$$

For the line segment

$$z(t)=-1+2t \qquad (0\le t\le 1) \tag{8}$$

For the right circle, we have

$$z(t)=2+e^{it} \qquad (0\le t\le 2\pi) \tag{9}$$

We would like to combine these three curves using a single parameter in the range $0\le t\le 1$. A natural way to do this is use $0\le t\le \frac{1}{3}$ for the left circle, $\frac{1}{3}\le t\le \frac{2}{3}$ for the line segment, and $\frac{2}{3}\le t\le 1$ for the right circle. We can do this using 2.

For the left circle, $a=0$, $b=2\pi$, $c=0$ and $d=\frac{1}{3}$, so we have

$$z(t)=-2+e^{-6\pi it} \qquad \left(0\le t\le \frac{1}{3}\right) \tag{10}$$

For the line segment, $a=0$, $b=1$, $c=\frac{1}{3}$ and $d=\frac{2}{3}$, so we have

$$z(t) = -1 + 2(3t-1) \qquad \left(\frac{1}{3} \le t \le \frac{2}{3}\right) \tag{11}$$

For the right circle, $a = 0$, $b = 2\pi$, $c = \frac{2}{3}$ and $d = 1$, so we have

$$z(t) = 2 + e^{i(6\pi t - 4\pi)} \qquad \left(\frac{2}{3} \le t \le 1\right) \tag{12}$$

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Pingback: Continuously deformable loops