CONVERGENCE OF TAYLOR SERIES

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Here are a few examples of Taylor series, with the area of convergence.

Example 1. $f(z)=\frac{1}{1+z}$ about $z_0=0$.

The derivatives are

$$\begin{aligned}
f(0) &= 1\\
f'(0) &= -\frac{1}{(1+z_0)^2}=-1\\
f''(0) &= \frac{2}{(1+z_0)^3}=2\\
&\vdots\\
f^{(n)}(0) &= \frac{(-1)^n n!}{(1+z_0)^{n+1}}=(-1)^n n!
\end{aligned}\tag{1}$$

The series is then

$$\frac{1}{1+z}=\sum_{j=0}^{\infty}\frac{(-1)^j j!}{j!}z^j=\sum_{j=0}^{\infty}(-1)^j z^j \tag{2}$$

The series converges for the disk $|z|<1$ since we must avoid the point $z=-1$.

Example 2. $f(z)=e^{-z^2}$ about $z_0=0$. Rather than calculating derivatives, it's easiest to use the series expansion of the exponential. We have

$$e^{-z^2}=\sum_{j=0}^{\infty}\frac{\left(-z^2\right)^j}{j!} \tag{3}$$

$$=1-z^2+\frac{z^4}{2}-\frac{z^6}{6}+\ldots \tag{4}$$

Since the exponential is an entire function, this is valid for the entire complex plane.

Example 3. $f(z)=z^3\sin(3z)$ about $z_0=0$. This is z^3 multiplied by the series expansion for the sine, so we have

$$\sin z=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^{2j-1}}{(2j-1)!} \tag{5}$$

which gives

$$z^3\sin(3z)=z^3\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(3z)^{2j-1}}{(2j-1)!} \tag{6}$$

$$=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}3^{2j-1}z^{2j+2}}{(2j-1)!} \tag{7}$$

$$=3z^4-\frac{9}{2}z^6+\frac{81}{40}z^8+\ldots \tag{8}$$

Both factors are entire, so the series is valid for the entire complex plane.

Example 4. $f(z)=2\cos z-ie^z$ about $z_0=0$. Here, we can subtract the series for the two terms. We have

$$\cos z=\sum_{j=0}^{\infty}\frac{(-1)^j z^{2j}}{(2j)!} \tag{9}$$

$$e^z=\sum_{j=0}^{\infty}\frac{z^j}{j!} \tag{10}$$

so we get

$$2\cos z-ie^z=\sum_{j=0}^{\infty}\left[\frac{(-1)^j 2z^{2j}}{(2j)!}-\frac{iz^j}{j!}\right] \tag{11}$$

$$=2-i-iz+\left(\frac{i}{2}-1\right)z^2-\frac{i}{6}z^3+\ldots \tag{12}$$

Both terms are entire, so the series is valid for the entire complex plane.

Example 5. $f(z)=\frac{1+z}{1-z}$ about $z_0=i$. Using Maple to calculate the first few derivatives, we find

$$\begin{aligned} f(z) &= \frac{1+z}{1-z} \\ f'(z) &= \frac{2}{(1-z)^2} \\ f''(z) &= \frac{4}{(1-z)^3} \\ f^{(3)}(z) &= \frac{12}{(1-z)^4} \\ f^{(4)}(z) &= \frac{48}{(1-z)^5} \\ &\vdots \\ f^{(n)}(z) &= \frac{2n!}{(1-z)^{n+1}} \end{aligned} \tag{13}$$

so for $n \geq 1$

$$f^{(n)}(i) = \frac{2n!}{(1-i)^{n+1}} \tag{14}$$

and

$$f(i) = \frac{1+i}{1-i} = \frac{1}{2}(1+i)^2 = i \tag{15}$$

and the series is

$$\frac{1+z}{1-z} = \sum_{j=0}^{\infty} \frac{2}{(1-i)^{j+1}} (z-i)^j \tag{16}$$

$$= i + i(z-i) + \left(-\frac{1}{2} + \frac{i}{2}\right)(z-i)^2 - \frac{1}{2}(z-i)^3 + \ldots \tag{17}$$

The series is valid for the disk centred at $z_0 = i$ and extending as far as $z = 1$, which is $|z-i| < \sqrt{2}$.

Example 6. $f(z) = \cos z$ about $z_0 = \frac{\pi}{4}$. The derivatives are

$$\begin{aligned}
f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\
f'\left(\frac{\pi}{4}\right) &= -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \\
f''\left(\frac{\pi}{4}\right) &= -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \\
f^{(3)}\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}
\end{aligned} \tag{18}$$

with the pattern repeating in groups of four hereafter. The series is

$$\cos z = \sum_{j=0}^{\infty} \frac{f^{(j)}\left(\frac{\pi}{4}\right)}{j!}\left(z-\frac{\pi}{4}\right)^j \tag{19}$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{2}\left(z-\frac{\pi}{4}\right) - \frac{1}{4}\sqrt{2}\left(z-\frac{\pi}{4}\right)^2 + \frac{1}{12}\sqrt{2}\left(z-\frac{\pi}{4}\right)^3 + \ldots \tag{20}$$

Example 7. $f(z) = z/(1-z)^2$ about $z_0 = 0$. The first few derivatives are

$$\begin{aligned}
f(z) &= \frac{z}{(1-z)^2} \\
f'(z) &= \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} \\
f''(z) &= \frac{4}{(1-z)^3} + \frac{6z}{(1-z)^4} \\
f^{(3)}(z) &= \frac{18}{(1-z)^4} + \frac{24z}{(1-z)^5} \\
&\vdots \\
f^{(n)}(z) &= \frac{n(n!)}{(1-z)^{n+1}} + \frac{(n+1)!z}{(1-z)^{n+2}}
\end{aligned} \tag{21}$$

The last formula can be proved by induction. Starting with $f^{(n)}$ we take the next derivative:

$$f^{(n+1)}(z) = \frac{n(n+1)(n!)}{(1-z)^{n+2}} + \frac{(n+1)!}{(1-z)^{n+2}} + \frac{(n+2)(n+1)!z}{(1-z)^{n+3}} \tag{22}$$

$$= \frac{(n+1)(n+1)!}{(1-z)^{n+2}} + \frac{(n+2)!z}{(1-z)^{n+3}} \tag{23}$$

Therefore

$$f^{(n)}(0) = n(n!) \tag{24}$$

and the series is

$$\frac{z}{(1-z)^2} = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = \sum_{j=0}^{\infty} j z^j \tag{25}$$

This converges in the disk $|z| < 1$.