CONVERGENT SERIES

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A *series* is the sum of a number of terms over either a finite or infinite range. A common series is the geometric series, defined by

$$S=\sum_{j=0}^{\infty}a^{j} \tag{1}$$

where a is a constant. We can find a general formula for this series as follows. We have, for some finite integer n

$$S_{n}=1+a+a^{2}+\ldots+a^{n} \tag{2}$$

$$aS_{n}=a+a^{2}+\ldots+a^{n}+a^{n+1} \tag{3}$$

Subtracting the second equation from the first we have

$$S_{n}\left(1-a\right)=1-a^{n+1} \tag{4}$$

so

$$S_{n}=\frac{1-a^{n+1}}{1-a} \tag{5}$$

If $|a|<1$ then we can let $n\to\infty$, by which the term $a^{n+1}\to0$ and we have

$$S_{n}\to S=\frac{1}{1-a} \tag{6}$$

Example 1. Find

$$S=\sum_{j=0}^{\infty}\left(\frac{i}{3}\right)^{j} \tag{7}$$

Here $a=i/3$ and $|a|=1/3<1$ so

$$S = \frac{1}{1-i/3} \tag{8}$$
$$= \frac{1+i/3}{(1-i/3)(1+i/3)} \tag{9}$$
$$= \frac{1+i/3}{10/9} \tag{10}$$
$$= \frac{9}{10}+\frac{3}{10}i \tag{11}$$

Example 2. Find

$$S=\sum_{j=0}^{\infty}\frac{3}{(1+i)^j} \tag{12}$$

Here the factor 3 is not raised to any power so we can take it outside the sum, giving

$$S=3\sum_{j=0}^{\infty}\frac{1}{(1+i)^j} \tag{13}$$
$$= \frac{3}{1-1/(1+i)} \tag{14}$$
$$= \frac{3(1+i)}{1+i-1} \tag{15}$$
$$= 3-3i \tag{16}$$

Example 3. Find

$$S=\sum_{j=0}^{\infty}(-1)^j\left(\frac{2}{3}\right)^j \tag{17}$$

We have

$$S=\frac{1}{1+2/3} \tag{18}$$
$$= \frac{3}{5} \tag{19}$$

Example 4. Find

$$S_p=\sum_{j=14}^{\infty}\frac{1}{(2i)^j} \tag{20}$$

Because the lower limit is not zero, we can treat this as the difference $S - S_{13}$ and use 5 for S_{13}. We have

$$S = \frac{1}{1 - 1/2i} \tag{21}$$

$$= \frac{2i}{2i - 1} \tag{22}$$

$$= \frac{2i(-2i-1)}{(2i-1)(-2i-1)} \tag{23}$$

$$= \frac{4}{5} - \frac{2}{5}i \tag{24}$$

Also, using Maple to do the arithmetic

$$S_{13} = \frac{1 - (1/2i)^{14}}{1 - 1/2i} \tag{25}$$

$$= \left(\frac{4}{5} - \frac{2}{5}i\right)\left(1 - \frac{1}{(2i)^{14}}\right) \tag{26}$$

$$= \left(\frac{4}{5} - \frac{2}{5}i\right)\left(1 + \frac{1}{16384}\right) \tag{27}$$

$$= \frac{3277}{4096} - \frac{3277}{8192}i \tag{28}$$

The sum S_p is therefore

$$S_p = S - S_{13} = -\frac{1}{20480} + \frac{i}{40960} \tag{29}$$

Example 5. Find

$$S = \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^{2j} \tag{30}$$

This is the same as

$$S = \sum_{j=0}^{\infty} \left(\frac{1}{9}\right)^{j} = \frac{1}{1 - 1/9} = \frac{9}{8} \tag{31}$$

Example 6. (Not a geometric series.) Find

$$S = \sum_{j=0}^{\infty} \left[\frac{1}{j+2} - \frac{1}{j+1}\right] \tag{32}$$

We can write out the first few terms to see that

$$S=\frac{1}{2}-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{4}-\frac{1}{3}+\ldots \tag{33}$$

Thus all terms except for the -1 cancel in pairs, and the magnitude of

$$\left|\frac{1}{j+2}-\frac{1}{j+1}\right|=\left|\frac{-1}{(j+2)(j+1)}\right| \tag{34}$$

goes to zero for large j, so we have

$$S=-1 \tag{35}$$

You have to be a bit careful with series like this. If we tried to solve it by working out $\sum\frac{1}{j+2}$ and $\sum\frac{1}{j+1}$ separately, we would find the both of these series diverge, so we'd be trying to find the difference between two infinities, which is undefined. As another example, if we tried to find the sum of $1-1+1-1+1-\ldots$ we might be tempted to say it is zero because each +1 is cancelled by a -1. However, in this case, the limit of the terms does not go to zero, so the sum is divergent.

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