CURL AND STOKES'S THEOREM

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One of the main vector derivatives is known as the *curl*. To see the motivation for it, we will consider a vector field $\mathbf{A}$ which, to make things concrete, can represent the flow of a fluid. We can imagine putting a little paddle wheel into the fluid at various points to measure the circulation of the fluid. If there is a vortex in the fluid and we put a paddle wheel centred around the vortex, the wheel will turn. In general, any difference in tangential speed of the fluid from one side of the wheel to the other would cause the wheel to turn. For example, in a river, frequently the water flows more rapidly in the middle of the stream than it does at the banks, so if we put a paddle wheel (with a vertical axis, not a horizontal axis such as would be found in a water mill) anywhere in the river between the middle and the bank, the difference in speeds of the water across the wheel would cause it to turn.

As always, in calculus, we're interested in how these ideas work on an infinitesimal scale. To keep things simple, we'll consider only one component of the circulation: that in the xy plane. Suppose we have a small rectangle in this plane with one corner at location (x_0, y_0) and the diagonally opposite corner at $(x_0 + dx, y_0 + dy)$. To calculate the circulation around this rectangle, we would need to integrate the component of $\mathbf{A}$ that is parallel to each edge, with the integral extending around all four sides of the rectangle. That is, we would need to integrate $\mathbf{A}\cdot d\mathbf{l}$ around the rectangle, where $d\mathbf{l}$ is the incremental length vector that points along an edge. The direction in which we go around the rectangle will affect the sign of the answer, and by convention the integral is taken counterclockwise. That is, along the bottom edge $d\mathbf{l} = dx\hat{\mathbf{i}}$ and along the top edge, $d\mathbf{l} = -dx\hat{\mathbf{i}}$, and along the left edge $d\mathbf{l} = -dy\hat{\mathbf{j}}$, and right edge $d\mathbf{l} = dy\hat{\mathbf{j}}$.

Taking the dot product of $\mathbf{A}$ with $d\mathbf{l}$ selects out the x component of $\mathbf{A}$ along the horizontal paths, and the y component of $\mathbf{A}$ along the vertical paths. Since the components of $\mathbf{A}$ are functions of x and y, we can use a first order expansion to get their values on all four sides of the rectangle.

At location (x_0, y_0), we have $A_x = A_x(x_0, y_0)$, while at $(x_0, y_0 + dy)$ we get $A_x = A_x(x_0, y_0 + dy) = A_x(x_0, y_0) + (\partial A_x/\partial y)(dy)$ to first order. So to first order, the contribution to the circulation integral from the top and bottom edges is

$$\underbrace{A_x(x_0,y_0)dx}_{\text{bottom}}+\underbrace{\left[A_x(x_0,y_0)+\frac{\partial A_x}{\partial y}dy\right](-dx)}_{\text{top}}=-\frac{\partial A_x}{\partial y}dx\;dy \tag{1}$$

By a similar argument we can work out the contributions from the left and right edges and get

$$\underbrace{A_y(x_0,y_0)(-dy)}_{\text{left}}+\underbrace{\left[A_y(x_0,y_0)+\frac{\partial A_y}{\partial x}dx\right]dy}_{\text{right}}=\frac{\partial A_y}{\partial x}dx\;dy \tag{2}$$

Combining the two, the total circulation is

$$\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)dx\;dy \tag{3}$$

Dividing out the infinitesimal area $dx\;dy$ we get the circulation per unit area as

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y} \tag{4}$$

We can do a similar analysis in the other two coordinate planes, and we find that the three terms are the components of the *curl* of **A** which is defined as

$$\text{curl}\,\boldsymbol{A}\equiv\nabla\times\mathbf{A}\equiv\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} \tag{5}$$

That is, the curl is defined analogously to a vector cross product, except the first 'vector' is the differential operator ∇.

Stokes's theorem can now be derived by considering some surface that is bounded by a closed path. Note that the surface itself is not closed (that is, it is not a surface like a sphere that has no edges), but apart from that, the exact form of the surface doesn't matter. All that matters is the boundary curve.

We can divide the surface up into a number of little patches and perform the analysis above on each patch, then add up the contribution of each patch to the circulaton to get the total circulation around the bounding edge. The key point is that each interior edge (that is, an edge of a path that does not lie along the edge of the surface) of one patch that is shared with a neighbouring patch will be traversed in opposite directions as we do the integrals over the two patches, so the contributions of all these internal edges cancel out, leaving only the unmatched edges along the boundary. The net result is that the integral of the curl of the vector field over the surface is equal to the line integral of the vector field around the boundary. That is

$$\boxed{\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int_L \mathbf{A}\cdot d\mathbf{l}} \tag{6}$$

where it is important to note that the integral on the left is taken over the entire area of the surface, while that on the right is taken over the surface's bounding curve. This is Stokes's theorem.

There are several theorems involving the curl, but probably the most important one is that the curl of a gradient of a scalar field V is always zero. That is

$$\nabla\times(\nabla V) = 0 \tag{7}$$

This can be seen most easily by substituting the gradient into the definition of the curl above:

$$\nabla\times(\nabla V) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} \tag{8}$$

Since the order in which partial derivatives are calculated doesn't matter (for any function of interest to physicists, anyway), this determinant is always zero, as you can verify by writing it out. For example, the $\hat{\boldsymbol{\imath}}$ component is

$$\left(\frac{\partial}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V}{\partial y}\right)\hat{\boldsymbol{\imath}} = 0 \tag{9}$$

A direct consequence of this in electrostatics is that, since the electric field can be expressed as the gradient of a potential function

$$\mathbf{E} = -\nabla V \tag{10}$$

the curl of the electric field (in electrostatics) is always zero:

$$\nabla\times\mathbf{E} = 0 \tag{11}$$

This is *not* true if we also have a time-varying magnetic field $\boldsymbol{B}$. In that case, we have Faraday's law of induction (one of Maxwell's equations):

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{12}$$

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