DERIVATIVES OF THE DELTA FUNCTION

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As the Dirac delta function is essentially an infinitely high spike at a single point, it may seem odd that its derivatives can be defined. The derivatives are defined using the delta function's integral property

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0) \tag{1}$$

Consider the integral involving the nth derivative $\delta^{(n)}(x)$ and apply integration by parts:

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x)\,dx = f(x)\,\delta^{(n-1)}(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta^{(n-1)}(x)\,dx \tag{2}$$

The integrated term is taken to be zero, since the delta function itself is constant (at zero) for all $x \neq 0$, so all its derivatives are zero except at $x = 0$. Therefore $\delta^{(n-1)}(x) = 0$ at the limits $-\infty$ and ∞. We're therefore left with

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\,\delta^{(n-1)}(x)\,dx \tag{3}$$

Since this is true for all functions $f(x)$, the integrands must be equal, so we get

$$f(x)\,\delta^{(n)}(x) = -f'(x)\,\delta^{(n-1)}(x) \tag{4}$$

A common case is the first derivative, which satisfies

$$f(x)\,\delta'(x) = -f'(x)\,\delta(x) \tag{5}$$

If $f(x) = x$, we get the relation

$$x\delta'(x) = -\delta(x) \tag{6}$$

By iterating 4, we get

$$f(x)\,\delta^{(n)}(x) = (-1)^n\,\delta(x)\,f^{(n)}(x) \tag{7}$$

Example 1. Suppose $f(x)=4x^2-1$. Then

$$\int_{-\infty}^{\infty}\left(4x^2-1\right)\delta'(x-3)\,dx = -\int_{-\infty}^{\infty}8x\delta(x-3)\,dx \tag{8}$$

$$= -24 \tag{9}$$

Example 2. With $f(x)=x^n$ we have, using 7

$$x^n\delta^{(n)}(x) = (-1)^n n!\delta(x) \tag{10}$$

Another use of the derivative of the delta function occurs frequently in quantum mechanics. In this case, we are faced with the integral

$$\int \delta'\left(x-x'\right) f\left(x'\right) dx' \tag{11}$$

where the prime in δ' refers to a derivative with respect to x, not x'. Thus the variable in the derivative is *not* the same as the variable being integrated over, unlike the preceding cases. In this case, since only x (and not x') is visible outside the integral, we can move the derivative outside the integral and get

$$\int \delta'\left(x-x'\right) f\left(x'\right) dx' = \frac{d}{dx}\int \delta\left(x-x'\right) f\left(x'\right) dx' \tag{12}$$

$$= f'(x) \tag{13}$$

Notice that in this case, there is no minus sign attached to the f' unlike in 5.

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