DIFFERENTIABLE COMPLEX FUNCTIONS

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A function of a real variable is *differentiable* if the limit

$$f'(x_0) \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1}$$

exists and is unique. The uniqueness requirement means that the same limit is obtained if we start above or below the point x_0. The value $f'(x_0)$ is the *derivative* of f at $x = x_0$.

A function $f(z)$ of a complex variable z is defined to be *differentiable* at a point z_0 if the limit

$$f'(z_0) \equiv \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{2}$$

exists and is unique. The catch here is that we're dealing with complex numbers which cover a plane rather than a line. Thus this limit must exist no matter how we approach z_0. We could approach it along the real axis, or along the imaginary axis, or along some convoluted path that converges on z_0. The definition 2 must give the same result no matter which path we take.

It turns out that, if f and g are differentiable at a point z, then the usual rules for derivatives from real calculus are valid for complex functions. The proofs of these would take us a bit far afield, so I'll just state them here.

$$(f \pm g)'(z) = f'(z) \pm g'(z) \tag{3}$$

$$(cf)'(z) = cf'(z) \text{ for constant } c \tag{4}$$

$$(fg)'(z) = f'(z)\,g(z) + f(z)\,g'(z) \quad \text{(product rule)} \tag{5}$$

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)\,g(z) - f(z)\,g'(z)}{g^2(z)} \quad \text{if } g(z) \neq 0 \text{ (quotient rule)} \tag{6}$$

$$\frac{d}{dz}(f(g(z))) = f'(g(z))\,g'(z) \quad \text{(chain rule)} \tag{7}$$

In particular, all polynomial functions are differentiable everywhere. Rational functions (a polynomial divided by another polynomial) are differentiable everywhere except where the denominator has roots.

What is perhaps a bit unexpected is that there are examples of functions of complex numbers that are not differentiable anywhere.

Example 1. The function $f(z) = \Re z$ is not differentiable. To see this, we apply the limit 2 first along a horizontal axis, where $\Delta z = \Delta x$. Then we get

$$\lim_{\Delta x \to 0} \frac{\Re(x + \Delta x + iy) - \Re(x + iy)}{\Delta x} = \frac{\Delta x}{\Delta x} = 1 \tag{8}$$

If we apply the limit along a vertical axis, then $\Delta z = i\Delta y$ and we get

$$\lim_{\Delta y \to 0} \frac{\Re(x + i(y + \Delta y)) - \Re(x + iy)}{i\Delta y} = \frac{x - x}{i\Delta y} = 0 \tag{9}$$

Thus the two limits are not equal, so the function is not differentiable.

Example 2. A similar argument applies to $f(z) = \Im z$. Again, we apply 2 along a horizontal axis:

$$\lim_{\Delta x \to 0} \frac{\Im(x + \Delta x + iy) - \Im(x + iy)}{\Delta x} = \frac{y - y}{\Delta x} = 0 \tag{10}$$

Along a vertical axis

$$\lim_{\Delta y \to 0} \frac{\Im(x + i(y + \Delta y)) - \Im(x + iy)}{i\Delta y} = \frac{\Delta y}{i\Delta y} = -i \tag{11}$$

Again, the two limits are not equal, so the function is not differentiable.

Example 3. The function $f(z) = |z|$ is also not differentiable. We have

$$f(z) = \sqrt{x^2 + y^2} \tag{12}$$

If we take the limit along a horizontal axis, then $\Delta z = \Delta x$ and we are effectively just taking the (real) derivative with respect to x which gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{(x + \Delta x)^2 + y^2} - \sqrt{x^2 + y^2}}{\Delta x} \tag{13}$$

$$= \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2}\right) \tag{14}$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \tag{15}$$

Taking the derivative along a vertical axis, then $\Delta z = i\Delta y$, and we take the derivative with respect to y, so we get

$$\lim_{\Delta y\to 0}\frac{f(x+i(y+\Delta y))-f(x+iy)}{i\Delta y} = \lim_{\Delta y\to 0}\frac{\sqrt{x^2+(y+\Delta y)^2}-\sqrt{x^2+y^2}}{i\Delta y} \tag{16}$$

$$= \frac{1}{i}\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) \tag{17}$$

$$= \frac{-iy}{\sqrt{x^2+y^2}} \tag{18}$$

The two limits are not the same so the function is not differentiable.

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