DIRAC DELTA FUNCTION - FOURIER TRANSFORM

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The Dirac delta function is defined by the two conditions

$$\delta(x) = 0 \text{ if } x \neq 0 \tag{1}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \tag{2}$$

The Fourier transform $\Delta(k)$ of $\delta(x)$ is:

$$\Delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x) e^{-ikx} dx \tag{3}$$

$$= \frac{1}{\sqrt{2\pi}} \tag{4}$$

The inverse of this relation is, using Plancherel's theorem

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Delta(k) e^{ikx} dk \tag{5}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk \tag{6}$$

This formula appears to make no sense as it stands, since e^{ikx} oscillates with the same amplitude over the entire infinite range of k. Yet this formula is used widely in physics, particularly in quantum mechanics and quantum field theory.

In fact, this formula can be made to look more credible by the following argument, due to Zee. Suppose we start with a finite integral:

$$d_K(x) \equiv \frac{1}{2\pi} \int_{-\frac{K}{2}}^{\frac{K}{2}} e^{ikx} dk \tag{7}$$

$$= \frac{1}{2\pi i x} e^{ikx} \Big|_{-\frac{K}{2}}^{\frac{K}{2}} \tag{8}$$

$$= \frac{1}{\pi x} \sin\left(\frac{Kx}{2}\right) \tag{9}$$

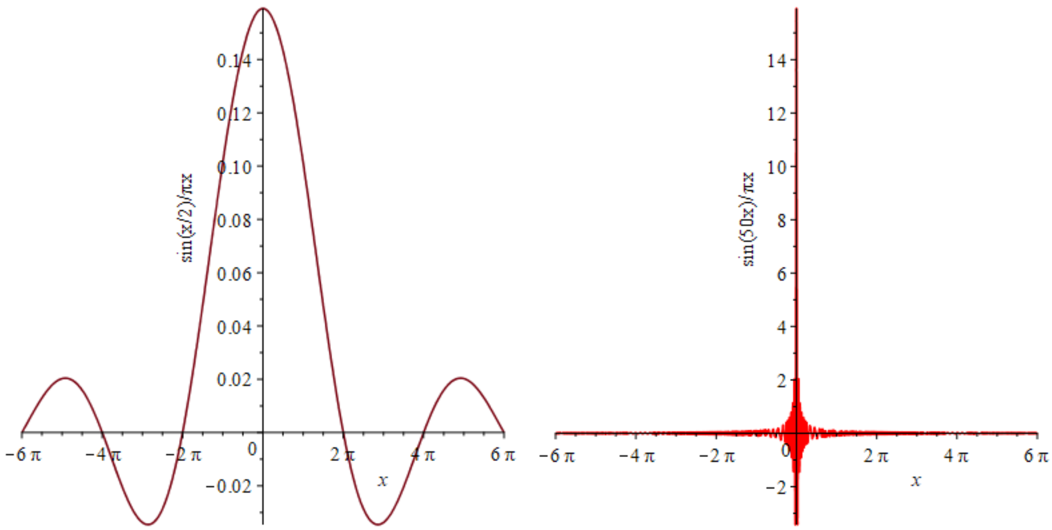

FIGURE 1. Plots of $\frac{1}{\pi x}\sin\left(\frac{Kx}{2}\right)$ for $K=1$ (left) and $K=100$ (right).

We can use the Taylor expansion to write

$$\frac{1}{\pi x}\sin\left(\frac{Kx}{2}\right)=\frac{1}{\pi x}\left(\left(\frac{Kx}{2}\right)-\frac{1}{3!}\left(\frac{Kx}{2}\right)^{3}+\ldots\right) \tag{10}$$

As $x\to0$, this has the limit

$$\lim_{x\to0}\frac{1}{\pi x}\sin\left(\frac{Kx}{2}\right)=\frac{K}{2\pi} \tag{11}$$

Thus as K increases, the function $\frac{1}{\pi x}\sin\left(\frac{Kx}{2}\right)$ has an increasing peak at $x=0$. It oscillates on either side of the origin with decreasing amplitude as you get farther from the origin. Also, $d_K(x)=0$ for the first time at $x=\pm\frac{2\pi}{K}$ on either side of the origin. A couple of plots (Fig. 1) show how the behaviour approaches a single peak at $x=0$ as K increases.

We see that for $K=100$ the plot has a strong spike at $x=0$ and falls to zero quite rapidly on either side.

Using the standard integral

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi \tag{12}$$

we find that

$$\int_{-\infty}^{\infty}d_K(x)\,dx=1 \tag{13}$$

Note that this result is independent of K, and remains true as $K \to \infty$. In this limit, the spike at $x = 0$ becomes infinitely large, and the width of the spike becomes infinitesimal. Thus we can define the delta function as this limit:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk \tag{14}$$

Another representation of $\delta(x)$ is the formula

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} \tag{15}$$

For any nonzero value of ϵ, this function has a peak of height $\frac{1}{\pi\epsilon}$ at $x = 0$, and its integral is

$$\begin{aligned} \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon\, dx}{x^2+\epsilon^2} &= \frac{1}{\pi}\arctan\left(\frac{x}{\epsilon}\right)\Big|_{-\infty}^{\infty} && (16) \\ &= 1 && (17) \end{aligned}$$

As $\epsilon \to 0$, the peak height at $x = 0$ becomes infinite, and the peak width becomes zero, since for any $x \neq 0$ $\delta(x) \to 0$ as $\epsilon \to 0$, so it satisfies the requirements of a delta function.

REFERENCES

(1) Anthony Zee, *Quantum Field Theory in a Nutshell*, 2nd edition (Princeton University Press, 2010) - Chapter I.2

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