

DIRAC DELTA FUNCTION AS LIMIT OF A GAUSSIAN INTEGRAL

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10.

Yet another form of the Dirac delta function is as the limit of a Gaussian integral. We start with

$$g_{\Delta}(x-x') = \frac{1}{(\pi\Delta^2)^{1/2}} e^{-(x-x')^2/\Delta^2} \quad (1)$$

If Δ^2 is real and positive, we have

$$\frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-x')^2/\Delta^2} dx = 1 \quad (2)$$

Thus the area under the curve is always 1, for any real value of Δ^2 . Now as $\Delta^2 \rightarrow 0$ the exponential becomes zero except when $x = x'$. The factor $1/(\pi\Delta^2)^{1/2}$ tends to infinity as $\Delta^2 \rightarrow 0$, but the exponential always tends to zero faster than any power of Δ , so $g_{\Delta}(x-x')$ tends to zero everywhere except at $x = x'$. Thus it satisfies the requirements of a delta function: it is zero everywhere except when $x - x' = 0$ and has an integral of 1. Thus

$$\lim_{\Delta \rightarrow 0} g_{\Delta}(x-x') = \delta(x-x') \quad (3)$$

However, if we plug the integral into Maple without any restrictions on Δ^2 , it informs us that the integral is still 1 even if Δ^2 is pure imaginary, provided that the imaginary number is positive, that is, we can write $\Delta^2 = i\beta^2$ for real β . Thus it would appear that g_{Δ} still gives a delta function in the limit $\Delta^2 \rightarrow 0$ even if Δ^2 is a positive imaginary number.

Shankar provides a rationale for this in his footnote to equation 1.10.19. In terms of β we can integrate some smooth function $f(x')$ multiplied by g_{Δ} over a region that includes $x' = x$.

$$\frac{1}{(\pi i\beta^2)^{1/2}} \int_{-\infty}^{\infty} e^{i(x-x')^2/\beta^2} f(x') dx \quad (4)$$

As $\beta^2 \rightarrow 0$, the exponent becomes a very large positive imaginary number everywhere except at $x = x'$, so the exponential oscillates very rapidly.

Provided that $f(x')$ doesn't vary as rapidly, the integral will contain equal positive and negative contributions everywhere except at $x = x'$ so in the limit of $\beta^2 = 0$, only the point $x = x'$ contributes, which means we can pull $f(x)$ out of the integral and get

$$\lim_{\beta^2 \rightarrow 0} \frac{1}{(\pi i \beta^2)^{1/2}} \int_{-\infty}^{\infty} e^{i(x-x')^2/\beta^2} f(x') dx = f(x) \quad (5)$$

Thus 3 is valid for all real Δ and for Δ^2 positive imaginary.

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