DIRAC DELTA FUNCTION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: Original: 16 Feb 2011, 14 Nov 2011, 6 Feb 2017; Revised 26 Dec 2020.

1. DIRAC DELTA FUNCTION IN ONE DIMENSION

One of the weirder bits of mathematics that the physics student will encounter is the Dirac delta function $\delta(x)$. In one dimension, the 'function' (technically it's really not a function at all, but a distribution) can be defined by saying $\delta(x)=0$ for all $x\neq 0$ but $\delta(x)=\infty$ at $x=0$. However, this definition isn't very satisfactory, and in fact doesn't define $\delta(x)$ uniquely. It is better to define it using two conditions:

$$\delta(x)=0 \text{ if } x\neq 0 \tag{1}$$

$$\int_{-\infty}^{\infty}\delta(x)dx=1 \tag{2}$$

It is the integral condition which pins the delta function down uniquely.

For those that like to visualize functions, the delta function can be thought of as a limit of a series of rectangular functions. If we define a rectangular function of width $1/w$ and height w (so its area is 1) centred horizontally on the y-axis and sitting with its base on the x-axis, then we can visualize the delta function as the limit of this rectangle as $w\to\infty$. As w gets larger, the rectangle gets taller and thinner, and in the limit it is an infinitely high spike with only an infinitesimal width, but with an area always equal to 1.

One consequence of the delta function being zero everywhere except at $x=0$ is that if we multiply it by any function, it doesn't matter what that function's values are except at $x=0$. That is, we can say

$$f(x)\delta(x)=f(0)\delta(x) \tag{3}$$

In terms of integrals, this means that

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx \tag{4}$$
$$= f(0)\int_{-\infty}^{\infty} \delta(x)dx \tag{5}$$
$$= f(0) \tag{6}$$

using the second defining property 2 of $\delta(x)$ above. The effect of integrating a function multiplied by the delta function is to pick out the function's value at $x = 0$.

It is easy enough to move the location of the delta function's spike. If we want the spike to appear at $x = a$ we can use the function $\delta(x-a)$, since the spike occurs when the delta function's argument is zero, that is, at $x-a = 0$. Thus we can generalize the integral formula above to

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{7}$$

One of the trickier formulas that causes some consternation amongst physics students is this formula:

$$\delta(kx) = \frac{1}{|k|}\delta(x) \tag{8}$$

where k is a non-zero constant.

Before we discuss what this means, we can run through the proof. Since the main use of delta functions is in integration, we can consider this formula in light of the integration condition. Suppose we define the variable transformation $y = kx$. Then $dy = k\,dx$. We'll take $k > 0$ first:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{1}{k}dy \tag{9}$$
$$= \frac{1}{k}\int_{-\infty}^{\infty} f(y/k)\delta(y)dy \tag{10}$$
$$= \frac{1}{k}f(0) \tag{11}$$

This is the same result as what we would get if we evaluated:

$$\frac{1}{k}\int_{-\infty}^{\infty} f(x)\delta(x)dx = \frac{1}{k}f(0) \tag{12}$$

so it seems reasonable to take $\delta(kx) = \delta(x)/k$ in this case.

If $k<0$,then the derivation is the same except that making the variable substitution $y=kx$ inverts the limits of integration, so we get

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{\infty}^{-\infty} f(y/k)\delta(y)\frac{1}{k}dy \tag{13}$$
$$= -\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{1}{k}dy \tag{14}$$
$$= -\frac{1}{k}f(0) \tag{15}$$
$$= \frac{1}{|k|}f(0) \tag{16}$$

Therefore, saying $\delta(kx)=\delta(x)/|k|$ covers both cases.

Now a problem many people have with this formula is this: if the definition of $\delta(x)$ is that it is zero everywhere, but infinite when $x=0$, then since the same can be said of $\delta(kx)$, why can't we just say $\delta(kx)=\delta(x)$?

The reason arises in the ambiguity of this non-integral definition of the delta function that I mentioned at the start. There are many way we can define a function that is zero everywhere but infinite at $x=0$. Look at it this way. Instead of using the limit of the sequence of rectangles that I did at the start, suppose we use a sequence of rectangles of width $1/kw$ and height w, so that their areas are all $1/k$. Now in the limit as $w\to\infty$ we get another function that is zero everywhere except at $x=0$ and has an infinite spike at $x=0$ but it is clearly not the same as $\delta(x)$ since the area of the spike is $1/k$ instead of 1. This new function which results from scaling the x axis using the transformation $x\to kx$ gives you a spike with an area of $1/k$ times the original.

A generalization of this formula is

$$\delta\left(f\left(x\right)-f\left(x_0\right)\right) = \frac{1}{\left|f'\left(x_0\right)\right|}\delta\left(x-x_0\right) \tag{17}$$

The proof of this follows the same lines as above. We consider the case $f'\left(x_0\right)>0$ and do the integral

$$\int_{-\infty}^{\infty} g(x)\delta\left(f\left(x\right)-f\left(x_0\right)\right)dx \tag{18}$$

where $g\left(x\right)$ is any function. Now use the substitution

$$y = f(x) - f(x_0) \tag{19}$$
$$dy = f'(x)\,dx \tag{20}$$
$$x = f^{-1}(y + f(x_0)) \tag{21}$$

where f^{-1} denotes an inverse function. Then we get

$$\int_{-\infty}^{\infty} g(x)\delta(f(x) - f(x_0))\,dx = \int_{-\infty}^{\infty} g\left(f^{-1}(y + f(x_0))\right)\frac{\delta(y)}{f'(x)}dy \tag{22}$$

Since $y = 0$ when $x = x_0$ we get

$$\int_{-\infty}^{\infty} g\left(f^{-1}(y + f(x_0))\right)\frac{\delta(y)}{f'(x)}dy = g\left(f^{-1}(f(x_0))\right)\frac{1}{f'(x_0)} \tag{23}$$
$$= \frac{g(x_0)}{f'(x_0)} \tag{24}$$

As above when $k < 0$, if $f'(x_0) < 0$ we get $-g(x_0)/f'(x_0)$ instead of 24 so the general formula is as given in 17.

In the even more general case where $f(x)$ has more than one zero, this formula generalizes to (I won't bother with the proof here):

$$\delta(f(x)) = \sum_i \frac{\delta(x - z_i)}{|f'(z_i)|} \tag{25}$$

where z_i are the zeroes of $f(x)$. Note in particular that $f'(z_i)$ is the *derivative* of f evaluated at points where the original function (*not* the derivative!) is zero.

Another formula that can cause nightmares is the derivative of the step function, that is of the function

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases} \tag{26}$$

Since the function is constant everywhere except at $x = 0$ its derivative is zero everywhere except at $x = 0$. However, the step function is discontinuous at this point, and since it jumps a finite amount over a single point, it would seem that its derivative is infinite at that point. To see what's going on, suppose we try the integral

$$\int_{-1}^{1} f(x)\frac{dH}{dx}dx = f(0)\int_{-1}^{1}\frac{dH}{dx}dx \quad (27)$$
$$= f(0)[H(1)-H(-1)] \quad (28)$$
$$= f(0) \quad (29)$$

(the limits on the integral could be any interval which includes 0). The first line uses the fact that $dH/dx = 0$ everywhere except $x = 0$, and the second line is just the ordinary evaluation of an integral.

Thus dH/dx satisfies both conditions of the delta function, so we can say

$$\frac{dH}{dx} = \delta(x) \quad (30)$$

Note that if the step is a different size, say $k > 0$, so that we have

$$H_k(x) = \begin{cases} 0 & x \le 0 \\ k & x > 0 \end{cases} \quad (31)$$

then the same analysis gives

$$\int_{-1}^{1} f(x)\frac{dH_k}{dx}dx = kf(0) \quad (32)$$

so from the earlier example, we get

$$\frac{dH_k}{dx} = \delta(x/k) \quad (33)$$
$$= k\delta(x) \quad (34)$$

2. Dirac delta function as limit of a Gaussian integral

Yet another form of the Dirac delta function is as the limit of a Gaussian integral. We start with

$$g_\Delta\left(x-x'\right) = \frac{1}{\left(\pi\Delta^2\right)^{1/2}} e^{-(x-x')^2/\Delta^2} \quad (35)$$

If Δ^2 is real and positive, we have

$$\frac{1}{\left(\pi\Delta^2\right)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-x')^2/\Delta^2}dx = 1 \quad (36)$$

[The Gaussian integral can be looked up in most tables of integrals, or evaluated using Maple.]

Thus the area under the curve is always 1, for any real value of Δ^2. Now as $\Delta^2 \to 0$ the exponential becomes zero except when $x = x'$. The factor $1/\left(\pi\Delta^2\right)^{1/2}$ tends to infinity as $\Delta^2 \to 0$, but the exponential always tends to zero faster than any power of Δ, so $g_\Delta(x - x')$ tends to zero everywhere except at $x = x'$. Thus it satisfies the requirements 1 and 2 of a delta function: it is zero everywhere except when $x - x' = 0$ and has an integral of 1. Thus

$$\lim_{\Delta \to 0} g_\Delta\left(x - x'\right) = \delta\left(x - x'\right) \tag{37}$$

However, if we plug the integral into Maple without any restrictions on Δ^2, it informs us that the integral is still 1 even if Δ^2 is pure imaginary, provided that the imaginary number is positive, that is, we can write $\Delta^2 = i\beta^2$ for real β. Thus it would appear that g_Δ still gives a delta function in the limit $\Delta^2 \to 0$ even if Δ^2 is a positive imaginary number.

Shankar provides a rationale for this in his footnote to equation 1.10.19. In terms of β we can integrate some smooth function $f(x')$ multiplied by g_Δ over a region that includes $x' = x$.

$$\frac{1}{\left(\pi i\beta^2\right)^{1/2}} \int_{-\infty}^{\infty} e^{i(x-x')^2/\beta^2} f\left(x'\right) dx \tag{38}$$

As $\beta^2 \to 0$, the exponent becomes a very large positive imaginary number everywhere except at $x = x'$, so the exponential oscillates very rapidly. Provided that $f(x')$ doesn't vary as rapidly, the integral will contain equal positive and negative contributions everywhere except at $x = x'$ so in the limit of $\beta^2 = 0$, only the point $x = x'$ contributes, which means we can pull $f(x)$ out of the integral and get

$$\lim_{\beta^2 \to 0} \frac{1}{\left(\pi i\beta^2\right)^{1/2}} \int_{-\infty}^{\infty} e^{i(x-x')^2/\beta^2} f\left(x'\right) dx = f(x) \tag{39}$$

Thus 37 is valid for all real Δ and for Δ^2 positive imaginary.

3. DIRAC DELTA FUNCTION IN THREE DIMENSIONS

The easiest way to define a three-dimensional delta function is just to take the product of three one-dimensional functions:

$$\delta_3(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z) \tag{40}$$

The integral of this function over any volume containing the origin is again 1, and the integral of any function of $\mathbf{r}$ is a simple extension of the one-dimensional case:

$$\int f(\mathbf{r})\delta_3(\mathbf{r}-\mathbf{a})d^3\mathbf{r} = f(\mathbf{a}) \tag{41}$$

In electrostatics, there is one situation where the delta function is needed to explain an apparent inconsistency involving the divergence theorem. If we have a point charge q at the origin, the electric field of that charge is (in SI units)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{42}$$

According to the divergence theorem, the surface integral of the field is equal to the volume integral of the divergence of that field:

$$\oint \mathbf{E}\cdot d\mathbf{a} = \int_V \nabla\cdot\mathbf{E}d^3\mathbf{r} \tag{43}$$

where the integral on the left is over some closed surface, and that on the right is over the volume enclosed by the surface. In electrostatics, the integral on the right evaluates to the total charge contained in the volume divided by ϵ_0

$$\int_V \nabla\cdot\mathbf{E}d^3\mathbf{r} = \frac{q}{\epsilon_0} \tag{44}$$

Now for the catch. If we calculate $\nabla\cdot\mathbf{E}$ (in spherical coordinates) for the point charge, we get, since only the radial component of the field is non-zero:

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right) && (45)\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial(1)}{\partial r} && (46)\end{aligned}$$

At this stage, we might be tempted to say that the derivative is zero (since the derivative of any constant is zero), but the problem is that at $r=0$ we also have a zero in the denominator, so we have the indeterminate fraction of zero-over-zero. Thus although it is true that $\nabla\cdot\mathbf{E}=0$ everywhere *except* the origin, we know from the divergence theorem that $\int_V \nabla\cdot\mathbf{E}d^3\mathbf{r} = \frac{q}{\epsilon_0}$ so we must have

$$\int_V \nabla\cdot\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)d^3\mathbf{r} = 4\pi \tag{47}$$

and

$$\nabla\cdot\left(\frac{1}{r^{2}}\hat{\mathbf{r}}\right)=0 \qquad \text{if} \quad r\neq0 \tag{48}$$

These two conditions can be satisfied if

$$\nabla\cdot\left(\frac{1}{r^{2}}\hat{\mathbf{r}}\right)=4\pi\delta_{3}(\mathbf{r}) \tag{49}$$

Another useful formula is

$$\begin{aligned}
\nabla\frac{1}{r} &= \nabla\frac{1}{\sqrt{x^{2}+y^{2}+z^{2}}} && (50)\\
&= -\frac{1}{\left(x^{2}+y^{2}+z^{2}\right)^{3/2}}\left[x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}\right] && (51)\\
&= -\frac{r\hat{\mathbf{r}}}{r^{3}} && (52)\\
&= -\frac{\hat{\mathbf{r}}}{r^{2}} && (53)
\end{aligned}$$

Therefore, the Laplacian of $\frac{1}{r}$ gives a delta function:

$$\nabla^{2}\frac{1}{r}=-4\pi\delta_{3}\left(\mathbf{r}\right) \tag{54}$$

One final property of the delta function is worth noting. Because of 2, the integral of $\delta\left(x\right)$ over x gives a dimensionless quantity. Thus, if x represents a physical distance, as it often does in physics, the delta function must have units of length^{-1} (or length^{-n} in n dimensions).

References

(1) Griffiths, David J. (2007) *Introduction to Electrodynamics*, 3rd Edition; Prentice Hall - Sec. 1.5.3
(2) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10.

Pingbacks

Pingback: Dirac delta function - Fourier transform
Pingback: Position operator - eigenfunctions
Pingback: Delta-function well - bound state
Pingback: Electric potential
Pingback: Ampère's law for steady currents
Pingback: Maxwell's equations
Pingback: Delta function well bound state - uncertainty principle

Pingback: Relativistic invariant integration measure
Pingback: Heat equation
Pingback: Potentials for a point charge
Pingback: Functionals and functional derivatives
Pingback: Maxwell's equations with varying charge but constant current
Pingback: Non-denumerable basis - position and momentum states
Pingback: Derivatives of the delta function
Pingback: Hydrogen atom - powers of the momentum operator
Pingback: Variational principle and the delta function well
Pingback: Quantum dots
Pingback: Integral form of the Schrödinger equation
Pingback: Current of a charged point particle