EQUALITY OF TWO RATIONAL FUNCTIONS

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Theorem 1. *Suppose we're given two rational functions in the form*

$$R_{m,n}(z) = \frac{P(z)}{Q(z)} \tag{1}$$

$$r_{m,n}(z) = \frac{p(z)}{q(z)} \tag{2}$$

The degrees m of the numerators P and p are the same, as are the degrees n of the denominators Q and q. If $R_{m,n}(z_k) = r_{m,n}(z_k)$ for $m+n+1$ different values of z_k, then the two rational functions are equal for all z.

Proof. We have

$$R_{m,n}(z_k) = r_{m,n}(z_k) \tag{3}$$

for the chosen values of z_k, which we can write as

$$\frac{P(z_k)}{Q(z_k)} = \frac{p(z_k)}{q(z_k)} \tag{4}$$

Cross multiplying, we have

$$P(z_k)\,q(z_k) = Q(z_k)\,p(z_k) \tag{5}$$

The combined polynomials $P(z_k)\,q(z_k)$ and $Q(z_k)\,p(z_k)$ both have degree $m+n$, and both contain $m+n+1$ coefficients, which are composed of combinations of the coefficients of the polynomials P,Q,p,q. If we generate $m+n+1$ linear equations by substituting in the $m+n+1$ values of z_k, we will have a system of $m+n+1$ linear equations in the $m+n+1$ coefficients of the compound polynomial. This system has a unique solution (unless two of the equations are linear multiples of each other, but we're assuming that doesn't happen), so the two compound polynomials Pq and Qp must be identical. Therefore

$$P(z)\,q(z) = Q(z)\,p(z) \tag{6}$$

or, dividing through by Qq

$$\frac{P(z)}{Q(z)} = \frac{p(z)}{q(z)} \tag{7}$$

□

Note that this doesn't guarantee that $P = p$ or $Q = q$, since there could be a common factor which cancels out. That is, we might have $P = \alpha p$ and $Q = \alpha q$ for some constant α. The key point is that α must be the same in both cases.

In practice, we would be given the coefficients of, say, P and Q and be asked to find the coefficients of p and q. That is, we would be given $m+n$ values for the coefficients of P and Q, and we would need to show that the m coefficients of p are multiples of the corresponding coefficients of P, and likewise that the n coefficients of q are the *same* multiples of the corresponding coefficients of Q.

Example 1. We're given the coefficients of $R_{1,1}$ and need to find the coefficients of $r_{1,1}$.

$$R_{1,1}(z) = \frac{a_0 + a_1 z}{b_0 + b_1 z} = \frac{1+2z}{3+4z} \tag{8}$$

$$r_{1,1}(z) = \frac{c_0 + c_1 z}{d_0 + d_1 z} \tag{9}$$

Cross multiplying gives us

$$(1+2z_k)(d_0 + d_1 z_k) = (3+4z_k)(c_0 + c_1 z_k) \tag{10}$$

We now choose $m+n+1 = 3$ values for the z_k. These are $z_1 = 1$, $z_2 = 2$ and $z_3 = 3$, which give the following system of equations:

$$\begin{aligned} 3d_0 + 3d_1 &= 7c_0 + 7c_1 \\ 5d_0 + 10d_1 &= 11c_0 + 22c_1 \\ 7d_0 + 21d_1 &= 15c_0 + 45c_1 \end{aligned} \tag{11}$$

Using Maple to solve this system giving values for c_0, c_1 and d_0 in terms of d_1, we have

$$\begin{aligned} c_0 &= \frac{d_1}{4} \\ c_1 &= \frac{d_1}{2} \\ d_0 &= \frac{3d_1}{4} \end{aligned} \tag{12}$$

We see from 8 that

$$\frac{c_0}{a_0} = \frac{c_1}{a_1} = \frac{d_1}{4} \tag{13}$$

$$\frac{d_0}{b_0} = \frac{d_1}{b_1} = \frac{d_1}{4} \tag{14}$$

Thus when we choose a value for d_1, $p = \frac{d_1}{4}P$ and $q = \frac{d_1}{4}Q$, so the two rational functions are the same, as the common factor of $\frac{d_1}{4}$ cancels out.

In general, we are faced with a system of $m+n+1$ linear equations in $m+n+2$ unknowns ($m+1$ for the coefficients of p and $n+1$ for the coefficients of q), so we can always solve for $m+n+1$ of the coefficients in terms of the remaining one.