ERROR ESTIMATES IN TAYLOR SERIES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 16 February 2025.

We can get estimates of the error in a truncated Taylor series by comparing the remainder with a geometric series.

Example 1. The series for e^z is

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{1}$$

For a truncated series, we define a truncated sum as

$$S_n \equiv \sum_{k=0}^{n} \frac{z^k}{k!} \tag{2}$$

which gives us

$$e^z - S_n = \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \equiv R_n \tag{3}$$

where R_n is the remainder. By factoring out the first term in R_n we have (strictly speaking, everything that follows should be inside modulus brackets, but to save clutter I've omitted them. That is, we're really talking about $|R_n|$ and so on)

$$R_n = \frac{z^{n+1}}{(n+1)!}\left[1 + \frac{z}{n+2} + \frac{z^2}{(n+2)(n+3)} + \ldots\right] \tag{4}$$

The series in brackets can be compared to a geometric series:

$$1 + \frac{z}{n+2} + \frac{z^2}{(n+2)(n+3)} + \ldots \le \sum_{k=0}^{\infty} \frac{z^k}{(n+2)^k} \tag{5}$$

This is valid since all terms of order 2 and higher are less than (or equal to, if $z = 0$) the corresponding terms in the geometric series. Thus we have

n	z	$e^z - S_n$	R_n
7	0.9	1.185×10^{-5}	2.79×10^{-5}
8	0.9	1.173×10^{-6}	3.06×10^{-6}

TABLE 1. Errors in Taylor series for e^z.

$$R_n \le \frac{z^{n+1}}{(n+1)!}\sum_{k=0}^{\infty}\frac{z^k}{(n+2)^k} \tag{6}$$

$$= \frac{z^{n+1}}{(n+1)!}\frac{1}{1-z/(n+2)} \tag{7}$$

$$= \frac{z^{n+1}}{(n+1)!}\frac{n+2}{n+2-z} \tag{8}$$

If we now consider $|z| < 1$, then we take the modulus

$$|R_n| \le \frac{|z|^{n+1}}{(n+1)!}\frac{n+2}{|n+2-z|} \tag{9}$$

The largest possible value of $|z|^{n+1}$ is 1, and the smallest value of $|n+2-z|$ is $n+2-1 = n+1$, so we have

$$|R_n| \le \frac{1}{(n+1)!}\frac{n+2}{n+1} \tag{10}$$

$$= \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}\right) \tag{11}$$

From 11, to compute e^z to within $\pm 10^{-5}$ for $|z| < 1$, we have

$$\begin{aligned} R_7 &= 2.79 \times 10^{-5} \\ R_8 &= 3.062 \times 10^{-6} \end{aligned} \tag{12}$$

Thus we'd need up to $n = 8$. Doing the calculations (using Maple), we find (Table 1).

For smaller values of z, the error is correspondingly smaller, as you'd expect.

Example 2. The series for the sine is

$$\sin z = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!} \tag{13}$$

The truncated series is

n	z	$\sin z - U_n$	T_n
2	0.9	-9.4×10^{-5}	2×10^{-4}
3	0.9	1.06×10^{-6}	2.781×10^{-6}
5	0.9	10^{-10}	1.614×10^{-10}

TABLE 2. Errors for $\sin z$.

$$U_n \equiv \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \tag{14}$$

$$\sin z - U_n = \sum_{k=n+1}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \equiv T_n \tag{15}$$

where T_n is now the remainder, or error term. Following a similar procedure to Example 1, we factor out the leading term in T_n:

$$T_n = \frac{z^{2n+3}}{(2n+3)!} \left[(-1)^{n+1} + \frac{(-1)^{n+2} z^2}{(2n+5)(2n+4)} + \frac{(-1)^{n+3} z^4}{(2n+7)(2n+6)(2n+5)(2n+4)} + \dots \right] \tag{16}$$

The modulus of each term in the brackets can be compared with a geometric series:

$$|T_n| \leq \frac{|z|^{2n+3}}{(2n+3)!} \sum_{k=0}^{\infty} \frac{|z|^{2k}}{[(2n+5)(2n+4)]^k} \tag{17}$$

$$= \frac{|z|^{2n+3}}{(2n+3)!} \frac{1}{1 - |z|^2 / [(2n+5)(2n+4)]} \tag{18}$$

$$= \frac{|z|^{2n+3}}{(2n+3)!} \frac{(2n+5)(2n+4)}{(2n+5)(2n+4) - |z|^2} \tag{19}$$

$$= \frac{|z|^{2n+3}}{(2n+3)!} \frac{4n^2 + 18n + 20}{4n^2 + 18n + 20 - |z|^2} \tag{20}$$

To find the upper bound, we have, for $|z| < 1$, $|z|^{2n+3} \leq 1$ in the numerator and $4n^2 + 18n + 20 - |z|^2 \geq 4n^2 + 18n + 20 - 1 = 4n^2 + 18n + 19$ in the denominator. Thus

$$|T_n| \leq \frac{1}{(2n+3)!} \left[\frac{4n^2 + 18n + 20}{4n^2 + 18n + 19} \right] \tag{21}$$

Some sample calculations are given in Table 2. In this case, we see that even only a couple of terms in the series gives a reasonable result.