ESTIMATING CONTOUR INTEGRALS

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A contour integral is the integral of a complex function $f(z)$ along some contour Γ, where Γ is in general a connected set of smooth curves. We can use the usual methods of integration for real variables when calculating contour integrals in those cases where an antiderivative of the integrand can be found. For more exotic integrands, we can often get an upper bound on the value of the magnitude of the integral.

Suppose we have some function $f(z)$ that we wish to integrate over a contour Γ. If we can find the maximum magnitude M taken by $f(z)$ along the contour, and we can also find the length $\ell(\Gamma)$ of the contour, then we have the condition:

$$\left|\int_{\Gamma} f(z)\,dz\right| \le M\ell(\Gamma) \tag{1}$$

This result seems intuitive, as we're taking the maximum value of the magnitude and multiplying it by the length of the contour. If you want a more detailed derivation, you can see Saff and Snider's book, section 4.2.

Example 1. For the contour of the circle $|z| = 3$ traversed once, find the maximum of

$$\left|\int_{\Gamma} \frac{dz}{z^2 - i}\right| \tag{2}$$

The length of the contour is $3 \times 2\pi = 6\pi$. To find the maximum of the integrand, we need the *minimum* of the denominator. If $|z| = 3$, then

$$z^2 = 9e^{2i\theta} \tag{3}$$

with $0 \le \theta < 2\pi$. The quantity $z^2 - i$ is thus a circle of radius 9 with a centre at i. The minimum of $\left|z^2 - i\right|$ is the point on this circle closest to the origin, which occurs at $z = -8$. Thus

$$\left|z^2 - i\right| \ge 8 \tag{4}$$

The bound on the integral is therefore

$$\left|\int_\Gamma \frac{dz}{z^2-i}\right| \leq \frac{6\pi}{8} = \frac{3\pi}{4} \tag{5}$$

Example 2. Along the contour of the straight line from $z = R > 0$ to $z = R + 2\pi i$, find the maximum of

$$\left|\int_\Gamma \frac{e^{3z}}{1+e^z} dz\right| \tag{6}$$

The maximum of the numerator is

$$\max\left|e^{3z}\right| = e^{3R} \tag{7}$$

The minimum of the denominator occurs at $e^z = e^{R+\pi i} = -e^R$ and is

$$\min|1+e^z| = \left|1-e^R\right| = e^R - 1 \tag{8}$$

where the last equality holds because $R > 0$ so $e^R > 1$.

The length of the contour is 2π so

$$\left|\int_\Gamma \frac{e^{3z}}{1+e^z} dz\right| \leq \frac{2\pi e^{3R}}{e^R - 1} \tag{9}$$

Example 3. Along the contour of the arc of the circle $|z| = 1$ in the first quadrant, find the maximum of

$$\left|\int_\Gamma \text{Log } z dz\right| \tag{10}$$

The principal branch of the logarithm is given by

$$\text{Log } z = \text{Log } |z| + i\text{Arg } z \tag{11}$$

where the argument lies in the interval $(-\pi, \pi]$. Since Log $|z| = \text{Log } 1 = 0$ and the argument in the first quadrant is the interval $\left[0, \frac{\pi}{2}\right]$, we have

$$|\text{Log } z| \leq \frac{\pi}{2} \tag{12}$$

The length of the arc is $2\pi/4 = \pi/2$, so

$$\left|\int_\Gamma \text{Log } z dz\right| \leq \frac{\pi^2}{4} \tag{13}$$

Example 4. Along the contour given by the line segment from $z = 0$ to $z = i$, find the maximum of

$$\left|\int_{\Gamma}e^{\sin z}dz\right| \tag{14}$$

The sine function is defined by

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i} \tag{15}$$

Along the contour, z is purely imaginary, so that $z=iy$ with $0\le y\le1$. Thus the sine is

$$\sin z=\frac{e^{-y}-e^{y}}{2i}=\frac{i}{2}\left(e^{y}-e^{-y}\right) \tag{16}$$

Thus the exponent in $e^{\sin z}$ is always purely imaginary (or zero) and thus

$$\left|e^{\sin z}\right|=1 \tag{17}$$

along the entire contour. The length of the contour is 1, so

$$\left|\int_{\Gamma}e^{\sin z}dz\right|\le1\times1=1 \tag{18}$$

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