EXPONENTIAL MAPPING

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The exponential of a complex number is given by Euler's equation

$$e^{z}=e^{x}\left(\cos y+i\sin y\right) \tag{1}$$

The domain of definition and the range are both the entire complex plane. Although e^{x} is restricted to positive real numbers, the cosine and sine can both take values between -1 and 1, so e^{z} can be any complex number.

Using this form we can map sets of points in the complex plane to other sets of points.

Example 1. The vertical line $\Re z=1$ is mapped as follows.

$$e^{z}=e^{1}\left(\cos y+i\sin y\right) \tag{2}$$

The magnitude of the RHS is a constant e and y can take any value, so this mapping is the circle of radius e.

Example 2. Map the horizontal line $\Im z=\frac{\pi}{4}$. We have

$$e^{z}=e^{x}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) \tag{3}$$

$$=\frac{\sqrt{2}}{2}e^{x}\left(1+i\right) \tag{4}$$

The magnitude can be any value greater than zero. The point $1+i$ lies on the line $y=x$, so this mapping is the line $y=x$ for $x>0$. Note that the e^{x} here is a factor in the magnitude of the complex number, rather than the curve $y=e^{x}$ that we would get when plotting the real function $f\left(x\right)=e^{x}$.

Example 3. Map the strip $0\le\Im z\le\frac{\pi}{4}$. The upper bound at $\Im z=\frac{\pi}{4}$ is the line $y=x$ for $x>0$ from Example 2. The lower bound is the mapping of the real axis $y=0$, so we get

$$e^{z}=e^{x}\left(\cos0+i\sin0\right) \tag{5}$$

$$=e^{x} \tag{6}$$

Thus the line $\Im z = 0$ maps to the positive real axis $x > 0$. The strip $0 \leq \Im z \leq \frac{\pi}{4}$ thus maps to the infinite triangular wedge with vertex at the origin (but not including the origin) and bounded by the lines $y = 0$ and $y = x$, both with $x > 0$.