EXPONENTIAL OF THE RECIPROCAL

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Post date: 7 March 2025.

The function

$$f(z)=e^{1/z} \tag{1}$$

has an *essential singularity* at $z=0$. An essential singularity is a point z_0 which satisfies the conditions

- $|f(z)|$ is neither bounded near z_0 nor goes to infinity as $z\to z_0$.
- $f(z)$ assumes every complex number (with possibly one exception) as a value in every neighbourhood of z_0.

To see that $e^{1/z}$ satisfies these conditions, we observe first that, as $z\to 0$ along the real axis, the function increases without bound (and does, in fact, tend to infinity). However, if $z\to 0$ along the imaginary axis, the exponent $1/z$ is always purely imaginary, so $|f(z)|=1$ at every point on the imaginary axis.

To show the second condition, we observe that the 'one exception' is $f(z)=0$, which is never attained. To show that all complex numbers are assumed by $f(z)$ in any neighbourhood, no matter how small, around $z_0=0$, we resort to logarithms.

Take an arbitrary complex number c. Then the logarithm can be written as

$$\log c=\text{Log}\,|c|+i\text{Arg}c+2k\pi i \tag{2}$$

Remember that the uppercase Log refers to the principal logarithm, and the uppercase Arg is the principal argument in the range $(-\pi,\pi]$. Thus by choosing k large enough, we can make $\log c$ as large as we like. Now suppose we choose our neighbourhood to be the circle of radius ε around $z_0=0$. Thus we can make $\log c>1/\varepsilon$, no matter how small ε is. If we define

$$w\equiv\log c \tag{3}$$

then

$$|w| > \frac{1}{\varepsilon} \tag{4}$$

Thus by taking

$$z = \frac{1}{w} \tag{5}$$

we have

$$|z| < \varepsilon \tag{6}$$

and

$$e^{1/z} = e^w = e^{\log c} = c \tag{7}$$

As we didn't assume anything about c (except that $c \neq 0$) we see that we can always find a value of z such that $e^{1/z} = c$, in any neighbourhood around the origin.

We can explore 1 a bit more by plotting its level curves. That is we examine plots of

$$\left|e^{1/z}\right| = s \tag{8}$$

for various values of s. Expanding the exponent, we have

$$\left|e^{1/z}\right| = \left|e^{1/(x+iy)}\right| \tag{9}$$

$$= \left|e^{(x-iy)/\left(x^2+y^2\right)}\right| \tag{10}$$

$$= e^{x/\left(x^2+y^2\right)} \tag{11}$$

since the modulus operation removes the imaginary part of the exponent.

Thus we want the level curves for

$$e^{x/\left(x^2+y^2\right)} = s \tag{12}$$

Since everything here is real, we can use ordinary logarithms to get

$$\frac{x}{x^2+y^2} = \ln s \tag{13}$$

we can rewrite this as

$$x^2 - \frac{x}{\ln s} + y^2 = 0 \tag{14}$$

This is actually the equation of a circle with its centre on the x axis, as we can see as follows. Such a circle has the equation

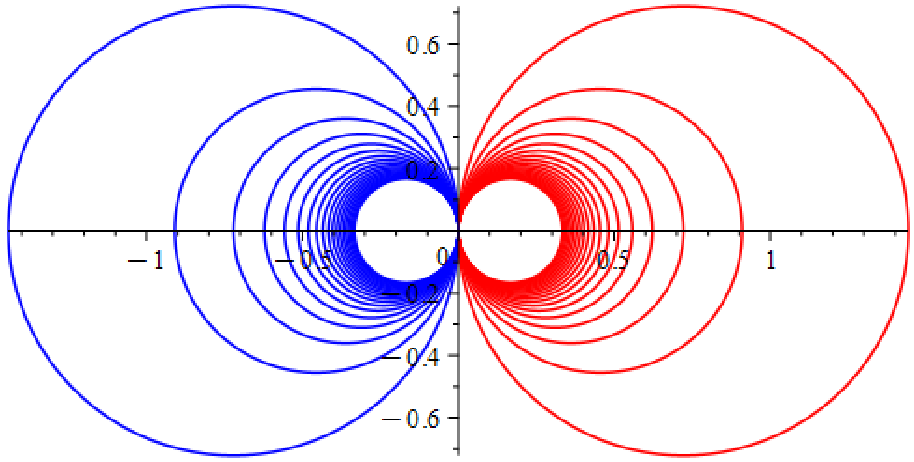

FIGURE 1. Level curves of $e^{1/z}$.

$$(x-a)^2+y^2=r^2 \tag{15}$$
$$x^2-2ax+a^2+y^2=r^2 \tag{16}$$

Comparing with 14, we can choose

$$\begin{aligned} a &= \frac{1}{2\ln s} \\ r &= a = \frac{1}{2\ln s} \end{aligned} \tag{17}$$

Thus the circles have the general equation

$$\left(x-\frac{1}{2\ln s}\right)^2+y^2=\frac{1}{4(\ln s)^2} \tag{18}$$

For $s>1$, the centre lies on the positive x axis and moves slowly towards the origin as s increases, while at the same time, the radius of the circle decreases. For $s<1$, $\ln s<0$ so the centre lies on the negative x axis and again moves slowly towards the origin, with decreasing radius. A plot of the first few curves is in Fig. 1.

The red curves represent $s>1$ (starting at $s=2$ and increasing by integer increments) and the blue curves $s<1$ (starting at $s=\frac{1}{2}$ and decreasing in the sequence $\frac{1}{n}$ thereafter). If $s=1$, $\ln s=0$ and the circle has an infinite radius.

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