FACTORING POLYNOMIALS

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Post date: 11 December 2024.

The *Fundamental Theorem of Algebra* states that every polynomial of nth degree with complex coefficients has n roots in $\mathbb{C}$, the set of complex numbers. The n roots need not be distinct, so it's possible that some of the roots occur two or more times. As a result, we can always factor a polynomial $p(z)$ into a product of linear factors. If $p(z)$ has degree n, then we can always write $p(z)$ as the product of k linear factors, as in

$$p(z) = (z - z_1)^{d_1} (z - z_2)^{d_2} \ldots (z - z_k)^{d_k} \tag{1}$$

where the z_i are the roots of p and

$$\sum_{m=1}^{k} d_m = n \tag{2}$$

A curious fact is that there are explicit formulas for finding the roots of polynomials of degree 2 (quadratics), 3 (cubics) and 4 (quartics), but it has been proved that no such formulas exist for polynomials of any higher degree.

Example 1. Factor $z^5 + (2+2i) z^4 + 2iz^3$. First, we can factor out z^3 which gives a double root of $z_1 = 0$. This leaves us with the quadratic

$$z^2 + (2+2i) z + 2i \tag{3}$$

We can use the quadratic formula to find the roots. We have

$$z_2 = \frac{-2 - 2i \pm \sqrt{(2+2i)^2 - 8i}}{2} \tag{4}$$

$$= -1 - i \tag{5}$$

Thus the remaining two roots are the same, and equal $-1-i$. The factorization is thus

$$z^5 + (2+2i) z^4 + 2iz^3 = z^3 (z+1+i)^2 \tag{6}$$

Example 2. Factor z^4-16. The four roots are the four fourth roots of 16, which are $-2,2,-2i,2i$ so the factorization is

$$z^4-16=(z+2)(z-2)(z+2i)(z-2i) \tag{7}$$

Example 3. Factoring the geometric series polynomial. Suppose we have the polynomial

$$g(z)=\sum_{k=0}^{n} z^k \tag{8}$$

This is a geometric series, and the sum is given by the formula

$$\sum_{k=0}^{n} z^k=\frac{z^{n+1}-1}{z-1} \tag{9}$$

The numerator on the RHS has as roots the $(n+1)$th roots of 1, so we have

$$z^{n+1}-1=(z-1)\left(z-e^{2\pi i/(n+1)}\right)\left(z-e^{4\pi i/(n+1)}\right)\ldots\left(z-e^{2n\pi i/(n+1)}\right) \tag{10}$$

Upon dividing by $z-1$ in 9 we eliminate the $z-1$ factor, so the remaining factors in 10 are the factors of $g(z)$.

For example, we can factor $g_6(z)=1+z+z^2+z^3+z^4+z^5+z^6$ as

$$g_6(z)=\left(z-e^{2\pi i/7}\right)\left(z-e^{4\pi i/7}\right)\left(z-e^{6\pi i/7}\right)\left(z-e^{8\pi i/7}\right)\left(z-e^{10\pi i/7}\right)\left(z-e^{12\pi i/7}\right) \tag{11}$$

Theorem 1. *Given the polynomial*

$$p(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_0 \tag{12}$$

and

$$|a_0|>1 \tag{13}$$

then $p(z)$ has at least one zero outside the unit circle in the complex plane.

Proof. Since the coefficient of the highest power z^n is $a_n=1$, we can write $p(z)$ in factored form as

$$p(z)=(z-z_1)(z-z_2)\ldots(z-z_n) \tag{14}$$

When these factors are multiplied out, the constant term a_0 in 12 is the product of the negatives of the roots. That is

$$a_0 = \prod_{k=1}^{n} (-z_k) \tag{15}$$

If $|a_0| > 1$ then we must have

$$|a_0| = \prod_{k=1}^{n} |z_k| > 1 \tag{16}$$

If all the z_k lay inside the unit circle, then $|z_k| < 1$ for all k, so their product would also be less than 1. Hence at least one of the z_k must lie outside the unit circle. □