FOURIER TRANSFORMS AND PLANCHEREL'S THEOREM

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A function $f(x)$ can be written as a Fourier transform in the form:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(k)e^{ikx}dk \tag{1}$$

This relation can be inverted by using *Plancherel's theorem*, which states

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx \tag{2}$$

Here we run through a plausibility argument which is a sort of physicist's proof of Plancherel's theorem. We start with Dirichlet's theorem which says that any (physically realistic, anyway) function $f(x)$ can be written as a Fourier series. We can show that this is equivalent to a series in complex exponentials. That is

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty}c_m e^{im\pi x/a} && (3)\\ &= \sum_{m=-\infty}^{\infty}c_m\left[\cos\frac{m\pi x}{a}+i\sin\frac{m\pi x}{a}\right] && (4)\\ &= c_0+\sum_{m=1}^{\infty}(c_m+c_{-m})\cos\frac{m\pi x}{a}+i\sum_{m=1}^{\infty}(c_m-c_{-m})\sin\frac{m\pi x}{a} && (5)\end{aligned}$$

We've used the facts that cosine is even and sine is odd. This is equivalent to a Fourier series:

$$f(x) = \sum_{m=0}^{\infty}\left[a_m\sin\frac{m\pi x}{a}+b_m\cos\frac{m\pi x}{a}\right] \tag{6}$$

where the coefficients are related by

$$\begin{aligned} b_0 &= c_0 && (7)\\ b_m &= c_m+c_{-m} && (8)\\ a_m &= i(c_m-c_{-m}) && (9)\end{aligned}$$

Inverting the relations we get, for $m>0$

$$\begin{aligned} c_m &= \frac{1}{2}\left(b_m - ia_m\right) && (10)\\ c_{-m} &= \frac{1}{2}\left(b_m + ia_m\right) && (11) \end{aligned}$$

We can get the coefficients in terms of $f(x)$ by integration. We multiply both sides of 3 by $\frac{1}{2a}e^{-in\pi x/a}$ and integrate over the interval $x\in[-a,a]$, using 3 to expand $f(x)$:

$$\frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx = \frac{1}{2a}\sum_{m=-\infty}^{\infty} c_m \int_{-a}^{a} e^{i(m-n)\pi x/a}dx \tag{12}$$

If $m\neq n$, the integral is

$$\int_{-a}^{a} e^{i(m-n)\pi x/a}dx = \frac{a}{i\left(m-n\right)\pi}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right] \tag{13}$$

Since $\sin\left(\left(m-n\right)\pi\right) = \sin\left(-\left(m-n\right)\pi\right)0$ and $\cos\left[-\left(m-n\right)\pi\right] = \cos\left[\left(m-n\right)\pi\right]$, the two exponentials cancel so the integral is zero if $m\neq n$.

If $m=n$, the integral is just

$$\int_{-a}^{a} dx = 2a \tag{14}$$

so the right hand side of 12 comes out to just c_n and we get

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx \tag{15}$$

Now we can make the substitutions

$$\begin{aligned} k &\equiv \frac{n\pi}{a} && (16)\\ F(k) &\equiv \sqrt{\frac{2}{\pi}}ac_n && (17) \end{aligned}$$

If Δk is the increment in k from one n to the next, then $\Delta k = \pi/a$. We can then write the original series 3 as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k)e^{ikx}\left(\frac{a}{\pi}\Delta k\right) && (18)\\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}F(k)e^{ikx}\Delta k && (19) \end{aligned}$$

The formula for c_n now becomes, using 17 and 15

$$c_n=\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k) = \frac{1}{2a}\int_{-a}^{a}f(x)e^{-ikx}dx \tag{20}$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}f(x)e^{-ikx}dx \tag{21}$$

Now we can take the limit as $a\to\infty$. In this case, $\Delta k\to dk$ (that is, it becomes a differential) and the sum becomes an integral, so we get from 19

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)e^{ikx}dk} \tag{22}$$

In the inverse formula 21, the limits on the integral become infinite, and we get the other half of Plancherel's theorem:

$$\boxed{F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx} \tag{23}$$

These last two formulas show that the Fourier transform of a function $f(x)$ can be inverted to get the function $F(k)$ as the inverse transform of the original function $f(x)$. The $\frac{1}{\sqrt{2\pi}}$ is a matter of convention. Sometimes the original transform is written as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{ikx}dk \tag{24}$$

and the inverse transform as

$$F(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx \tag{25}$$

Some textbooks use one convention and others the second convention, so you need to be sure of which convention is being used. In any case, the product of the two constants in front of the two integrals should always come out to $\frac{1}{2\pi}$.

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