FUNCTIONS OF A COMPLEX VARIABLE

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As with real variables, we can define a function $f(z)$ of a complex variable $z=x+iy$. The main difference between such a function and a function of a real variable is that $f(z)$ with typically consist of two real functions often written as u and v where

$$f(z)=w(x,y)=u(x,y)+iv(x,y) \tag{1}$$

The *domain of definition* of a function $f(z)$ is the set of all numbers z such that $f(z)$ is well-defined. This typically excludes such things as division by zero. Note that the domain of definition of a function is not necessarily the same thing as a domain in the complex plane. The domain of definition (often referred to simply as the 'domain', where the context makes it clear what we're talking about) can be a closed set or even just a few isolated points.

A function provides a *mapping* from its domain into another set called the *range*. Thus if a set D is the domain, it consists of all values of z for which $f(z)$ is defined, and the set R is the range, which consists of all values of w that result from applying the function to the domain.

Although the definition of $f(z)$ may look intricate, it is always possible to express it in the form 1, that is, as a sum of real and imaginary parts.

Example 1. $3z^2+5z+i+1$. We have

$$3z^2+5z+i+1=3(x+iy)^2+5x+5iy+i+1 \tag{2}$$

$$=3x^2-3y^2+5x+1+i(6xy+5y+1) \tag{3}$$

Thus

$$\begin{aligned} u(x,y)&=3x^2-3y^2+5x+1\\ v(x,y)&=6xy+5y+1 \end{aligned} \tag{4}$$

The domain is the entire complex plane.

Example 2. $1/z$.

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \tag{5}$$
$$= \frac{1}{x^2+y^2}(x-iy) \tag{6}$$
$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \tag{7}$$

The domain is the entire complex plane except for $z = 0$.

Example 3. $\frac{z+i}{z^2+1}$. This gets a bit messy so I used Maple to simplify the result.

$$\frac{z+i}{z^2+1} = \frac{(z+i)\left(\overline{z}^2+1\right)}{(z^2+1)\left(\overline{z}^2+1\right)} \tag{8}$$
$$= \frac{x}{x^2+y^2-2y+1} + i\frac{1-y}{x^2+y^2-2y+1} \tag{9}$$

The domain consists of all values except for $z = \pm i$.

Example 4. $\frac{2z^2+3}{|z-1|}$.

$$\frac{2z^2+3}{|z-1|} = \frac{2\left(x^2-y^2+2ixy\right)+3}{\sqrt{(x-1)^2+y^2}} \tag{10}$$
$$= \frac{2x^2-2y^2+3}{\sqrt{(x-1)^2+y^2}} + 4i\frac{xy}{\sqrt{(x-1)^2+y^2}} \tag{11}$$

The domain consists of all values except for $z = 1$.

Example 5. e^{3z}.

$$e^{3z} = e^{3x}e^{3iy} \tag{12}$$
$$= e^{3x}(\cos(3y) + i\sin(3y)) \tag{13}$$
$$= e^{3x}\cos(3y) + ie^{3x}\sin(3y) \tag{14}$$

The domain is the entire complex plane.

Example 6. $e^z + e^{-z}$.

$$e^z + e^{-z} = e^x(\cos y + i\sin y) + e^{-x}(\cos(-y) + i\sin(-y)) \tag{15}$$
$$= \left(e^x + e^{-x}\right)\cos y + i\left(e^x - e^{-x}\right)\sin y \tag{16}$$

The domain is the entire complex plane.