FUNCTIONS WITH ZEROES AND SINGULARITIES

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Here are some more examples of functions with zeroes and singularities.

Example 1. Consider

$$f(z) = \frac{1}{(2\cos z - 2 + z^2)^2} \tag{1}$$

We look at the denominator and expand in a series:

$$2\cos z - 2 + z^2 = 2 - z^2 + \frac{1}{12}z^4 - \frac{1}{360}z^6 + \mathcal{O}(z^8) - 2 + z^2 \tag{2}$$

$$= \frac{1}{12}z^4 - \frac{1}{360}z^6 + \mathcal{O}(z^8) \tag{3}$$

$$= z^4\left(\frac{1}{12} - \frac{z^2}{360}\right) + \mathcal{O}(z^8) \tag{4}$$

Thus the leading term is of order 4, so the reciprocal of the square of this has a pole of order 8 at $z = 0$.

Example 2. For the next examples, we look for a function with the given behaviour.

First, find a function with a zero of order 2 at $z = i$ and a pole of order 5 at $z = 2 - 3i$. This can be done with a rational function:

$$f(z) = \frac{z^2}{(z - 2 + 3i)^5} \tag{5}$$

Example 3. Find a function with a simple zero at $z = 0$ and an essential singularity at $z = 1$. The standard example of a function with an essential singularity is $e^{1/z}$, so we have

$$f(z) = ze^{1/(z-1)} \tag{6}$$

Example 4. Find a function with a removable singularity at $z=0$, a pole of order 6 at $z=1$ and an essential singularity at $z=i$. The function

$$\frac{\sin z}{z} = \frac{1}{z}\left(z-\frac{z^3}{3!}+\dots\right) \tag{7}$$

$$= 1-\frac{z^2}{3!}+\dots \tag{8}$$

has a removable singularity at $z=0$. Combining this with the other two requirements gives

$$f(z) = \frac{\sin z}{z(z-1)^6}e^{1/(z-i)} \tag{9}$$

Example 5. Find a function with a pole of order 2 at $z=1+i$ and essential singularities at $z=0$ and $z=1$. We have

$$f(z) = \frac{e^{1/z}e^{1/(z-1)}}{(z-1-i)^2} \tag{10}$$