GENERAL SERIES SUMMATION WITH RESIDUES

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We can extend the series summation technique to find sums of series with parameters in them. The formula is

$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{\ell} r_{\ell} \tag{1}$$

where r_{ℓ} is a residue of $g(z) = \pi f(z)\cot(\pi z)$ at a pole of $f(z)$. This assumes that $f(z)$ has no poles at integer real values.

In the following examples, a is a real non-integer constant.

Example 1. Find

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} \tag{2}$$

The function

$$f(z) = \frac{1}{(z+a)^2} \tag{3}$$

has a single pole of order 2 at $z=-a$, so we find the residue of

$$g(z) = \pi f(z)\cot(\pi z) \tag{4}$$

at this point. We use the formula

$$\operatorname{Res}(g; z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m g(z)\right] \tag{5}$$

with $m=2$ and $z_0=-a$. I've used Maple to find the derivatives and do the intervening algebra in these examples, though there's nothing beyond introductory calculus needed to do the calculations. We have

$$\operatorname{Res}(-a) = -\pi^2\left(1+\cot(\pi a)^2\right) \tag{6}$$

$$= -\pi^2\csc^2(\pi a) \tag{7}$$

so

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} = \pi^2 \csc^2(\pi a) \tag{8}$$

Example 2. Find

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} \tag{9}$$

We can write this as

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = \sum_{k=-\infty}^{\infty} \frac{1}{(k+ia)(k-ia)} \tag{10}$$

so we have simple poles at $k = \pm ia$. We need residues of

$$g(z) = \frac{\pi \cot(\pi z)}{(z+ia)(z-ia)} \tag{11}$$

at the poles. We get

$$\operatorname{Res}(g; ia) = -\frac{\pi i \cot(i\pi a)}{2a} \tag{12}$$

$$= -\frac{\pi \coth(\pi a)}{2a} \tag{13}$$

$$\operatorname{Res}(g; -ia) = -\frac{\pi i \cot(-i\pi a)}{-2a} \tag{14}$$

$$= -\frac{\pi \coth(\pi a)}{2a} \tag{15}$$

so we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi \coth(\pi a)}{a} \tag{16}$$

Example 3. Find

$$\sum_{k=-\infty}^{\infty} \frac{k^2-a^2}{(k^2+a^2)^2} = \sum_{k=-\infty}^{\infty} \frac{k^2-a^2}{(k+ia)^2(k-ia)^2} \tag{17}$$

This also has poles at $z = \pm ia$, but now they are of order 2, so we use 5 with $m = 2$. We have

$$g(z) = \frac{\pi \cot(\pi z)\left(z^2-a^2\right)}{(z+ia)^2(z-ia)^2} \tag{18}$$

Using Maple we get

$$\mathrm{Res}\,(g; z = ai) = \frac{2a\pi\coth(\pi a) + 2a^2\pi^2\left(\coth(\pi a)^2 - 1\right) - 2\pi a\coth(\pi a)}{4a^2} \tag{19}$$

$$= \frac{\pi^2\mathrm{csch}(\pi a)^2}{2} \tag{20}$$

$$\mathrm{Res}\,(g; z = -ai) = \frac{\pi^2\mathrm{csch}(\pi a)^2}{2} \tag{21}$$

so we have

$$\sum_{k=-\infty}^{\infty} \frac{k^2 - a^2}{(k^2 + a^2)^2} = -\pi^2\mathrm{csch}(\pi a)^2 \tag{22}$$

In the next examples, $0 < r < 1$.

Example 4. Find

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-r)^2 + a^2} \tag{23}$$

We have

$$g(z) = \frac{\pi\cot(\pi z)}{(z-r)^2 + a^2} \tag{24}$$

$$= \frac{\pi\cot(\pi z)}{(z - r - ia)(z - r + ia)} \tag{25}$$

This has simple poles at $z - r = \pm ia$ or $z = r \pm ia$. We have

$$\mathrm{Res}\,(g; r + ia) = \frac{-i\pi\cot(\pi(r + \mathrm{I}a))}{2a} \tag{26}$$

Using Maple, we can expand the cotangent to get

$$\mathrm{Res}\,(g; r + ia) = -\frac{i\pi}{2a}\left[\frac{\sin(\pi r)\cos(\pi r)}{\sin^2(\pi r) + \sinh^2(\pi a)} - \frac{i\sinh(\pi a)\cosh(\pi a)}{\sin^2(\pi r) + \sinh^2(\pi a)}\right] \tag{27}$$

The other residue is

$$\mathrm{Res}\,(g; r - ia) = \frac{-i\pi\cot(\pi(-r + \mathrm{I}a))}{2a} \tag{28}$$

$$= -\frac{i\pi}{2a}\left[-\frac{\sin(\pi r)\cos(\pi r)}{\sin^2(\pi r)+\sinh^2(\pi a)} - \frac{i\sinh(\pi a)\cosh(\pi a)}{\sin^2(\pi r)+\sinh^2(\pi a)}\right] \tag{29}$$

Adding 27 and 29, we see that the first term in brackets cancels out, leaving us with

$$\operatorname{Res}(g;r+ia)+\operatorname{Res}(g;r-ia) = -\frac{\pi}{a}\frac{\sinh(\pi a)\cosh(\pi a)}{\sin^2(\pi r)+\sinh^2(\pi a)} \tag{30}$$

$$= -\frac{\pi}{2a}\frac{\sinh 2\pi a}{\sin^2(\pi r)+\sinh^2(\pi a)} \tag{31}$$

The sum is

$$\sum_{k=-\infty}^{\infty}\frac{1}{(k-r)^2+a^2} = \frac{\pi}{2a}\frac{\sinh 2\pi a}{\sin^2(\pi r)+\sinh^2(\pi a)} \tag{32}$$

Example 5. Finally, we find

$$\sum_{k=-\infty}^{\infty}\frac{(k-r)^2-a^2}{\left[(k-r)^2+a^2\right]^2} = \sum_{k=-\infty}^{\infty}\frac{(k-r)^2-a^2}{(k-r-ia)^2(k-r+ia)^2} \tag{33}$$

With

$$g(z) = \pi\cot(\pi z)\frac{(z-r)^2-a^2}{(z-r-ia)^2(z-r+ia)^2} \tag{34}$$

we now have order 2 poles at $z = r \pm ia$. Using 5 gets quite messy, but using Maple to simplify the results we have

$$\operatorname{Res}(g;r+ia) = -\frac{\pi^2\csc(\pi(r+ia))^2}{2} \tag{35}$$

$$\operatorname{Res}(g;r-ia) = \frac{\pi^2\operatorname{csch}(\pi(ir+a))^2}{2} \tag{36}$$

Expanding these two residues we find that the imaginary parts cancel out in the sum, leaving us with

$$\operatorname{Res}(g;r+ia)+\operatorname{Res}(g;r-ia) = \frac{\pi^2\left(-\sin(\pi r)^2\cosh(\pi a)^2+\sinh(\pi a)^2\cos(\pi r)^2\right)}{\left(\sin(\pi r)^2+\sinh(\pi a)^2\right)^2} \tag{37}$$

Therefore the sum is

$$\sum_{k=-\infty}^{\infty} \frac{(k-r)^2 - a^2}{\left[(k-r)^2 + a^2\right]^2} = \frac{\pi^2 \left(\sin(\pi r)^2 \cosh(\pi a)^2 - \sinh(\pi a)^2 \cos(\pi r)^2\right)}{\left(\sin(\pi r)^2 + \sinh(\pi a)^2\right)^2} \tag{38}$$

As a footnote, the answer given in Saff and Snider's problem 17e in section 6.3 is

$$\sum_{k=-\infty}^{\infty} \frac{(k-r)^2 - a^2}{\left[(k-r)^2 + a^2\right]^2} = \frac{\pi^2 (1 - \cos(2\pi r)\cosh(2\pi a))}{2\left(\sin(\pi r)^2 + \sinh(\pi a)^2\right)^2} \tag{39}$$

To show that 38 and 39 are in fact the same requires a bit of juggling, but here goes. It's easiest to start with 39. To simplify the notation, we'll define

$$\begin{aligned} c &\equiv \cos\pi r \\ s &\equiv \sin\pi r \\ \xi &\equiv \cosh\pi a \\ \sigma &\equiv \sinh\pi a \end{aligned} \tag{40}$$

We also use the double angle formulas

$$\begin{aligned} \cos 2\pi r &= \cos^2 \pi r - \sin^2 \pi r = c^2 - s^2 \\ \cosh 2\pi a &= \cosh^2 \pi a + \sinh^2 \pi a = \xi^2 + \sigma^2 \end{aligned} \tag{41}$$

and the identities

$$\begin{aligned} c^2 + s^2 &= 1 \\ \xi^2 - \sigma^2 &= 1 \end{aligned} \tag{42}$$

The numerator in 39 is then

$$1 - \cos(2\pi r)\cosh(2\pi a) = 1 - \left(c^2 - s^2\right)\left(\xi^2 + \sigma^2\right) \tag{43}$$

$$= 1 - c^2\xi^2 - c^2\sigma^2 + s^2\xi^2 + s^2\sigma^2 \tag{44}$$

$$= 1 - \left(c^2 + c^2\sigma^2\right) - c^2\sigma^2 + s^2\xi^2 + s^2\xi^2 - s^2 \tag{45}$$

$$= 1 - \left(c^2 + s^2\right) - 2c^2\sigma^2 + 2s^2\xi^2 \tag{46}$$

$$= 2\left(s^2\xi^2 - c^2\sigma^2\right) \tag{47}$$

$$= 2\left(\sin^2 \pi r \cosh^2 \pi a - \cos^2 \pi r \sinh^2 \pi a\right) \quad (48)$$

Substituting this into 39 gives us 38, so the two forms are in fact identical.