GENERALIZED BINOMIAL THEOREM

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The binomial theorem is usually stated as the expansion of a binomial to an integer power n, as in

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \tag{1}$$

Using Taylor series we can write a generalized version of the binomial theorem by expanding $(1+z)^\alpha$ where α is any complex number. We write this as

$$f(z)=(1+z)^\alpha=e^{\alpha\operatorname{Log}(1+z)} \tag{2}$$

which is valid for $|z|<1$, since we must avoid the singularity at $z=1$ in the logarithm.

To get the Taylor series, we need a sequence of derivatives of $f(z)$ evaluated at $z_0=0$. We have (using Maple for the derivatives, although the derivatives are found using the chain rule if you want to do them by hand):

$$f(0)=e^{\alpha\operatorname{Log}(1)}=e^0=1 \tag{3}$$

$$f'(z)=\frac{\alpha e^{\operatorname{Log}(1+z)}}{1+z} \tag{4}$$

$$f'(0)=\alpha \tag{5}$$

$$f''(z)=-\frac{\alpha e^{\operatorname{Log}(1+z)}}{(1+z)^2}+\frac{\alpha^2 e^{\operatorname{Log}(1+z)}}{(1+z)^2} \tag{6}$$

$$f''(0)=-\alpha+\alpha^2=\alpha(\alpha-1) \tag{7}$$

$$f^{(3)}(z)=\frac{2\alpha e^{\operatorname{Log}(1+z)}}{(1+z)^3}-\frac{3\alpha^2 e^{\operatorname{Log}(1+z)}}{(1+z)^3}+\frac{\alpha^3 e^{\operatorname{Log}(1+z)}}{(1+z)^3} \tag{8}$$

$$f^{(3)}(0)=2\alpha-3\alpha^2+\alpha^3=\alpha(\alpha-1)(\alpha-2) \tag{9}$$

The first terms of the Taylor series are then

$$(1+z)^{\alpha} = 1 + \frac{\alpha}{1!}z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \ldots \tag{10}$$

The complete series is given by

$$(1+z)^{\alpha} = 1 + \sum_{j=1}^{\infty}\left[\prod_{k=0}^{j-1}(\alpha-k)\right]\frac{z^j}{j!} \tag{11}$$

We can show this by defining a generalized binomial coefficient as

$$\binom{\alpha}{n} \equiv \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \tag{12}$$

valid for any complex number α. Then we can write 11 as

$$f(z) = (1+z)^{\alpha} = \sum_{j=0}^{\infty}\binom{\alpha}{j}z^j \equiv g(z) \tag{13}$$

Now take successive derivatives of this equation. For the LHS, we have

$$f'(z) = \alpha(1+z)^{\alpha-1} \tag{14}$$
$$f'(0) = \alpha \tag{15}$$
$$\frac{f'(0)}{1!} = \binom{\alpha}{1} \tag{16}$$
$$f''(z) = \alpha(\alpha-1)(1+z)^{\alpha-2} \tag{17}$$
$$\frac{f''(0)}{2!} = \frac{\alpha(\alpha-1)}{2!} = \binom{\alpha}{2} \tag{18}$$
$$\vdots \tag{19}$$
$$f^{(n)}(z) = \alpha(\alpha-1)\ldots(\alpha-n+1)(1+z)^{\alpha-n} \tag{20}$$
$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} = \binom{\alpha}{n} \tag{21}$$

For the RHS of 13 we have

$$g'(z) = \sum_{j=1}^{\infty} \binom{\alpha}{j} j z^{j-1} \tag{22}$$

$$g'(0) = \binom{\alpha}{1} \tag{23}$$

$$g''(z) = \sum_{j=2}^{\infty} \binom{\alpha}{j} j(j-1) z^{j-2} \tag{24}$$

$$g''(0) = \binom{\alpha}{2} 2! \tag{25}$$

$$\frac{g''(0)}{2!} = \binom{\alpha}{2} \tag{26}$$

$$\vdots \tag{27}$$

$$g^{(n)}(z) = \sum_{j=n}^{\infty} \binom{\alpha}{j} n! z^{j-n} \tag{28}$$

$$\frac{g^{(n)}(0)}{n!} = \binom{\alpha}{n} \tag{29}$$

Thus

$$\frac{f^{(n)}(0)}{n!} = \frac{g^{(n)}(0)}{n!} \tag{30}$$

so the coefficients in the series represented by $g(z)$ match the corresponding coefficients for $f(z)$ so the two functions must be the same.