GENERALIZED L'HOPITAL'S RULE

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Post date: 21 February 2025.

L'Hôpital's ruleis usually given as the theorem:

If $f(z)$ and $g(z)$ are analytic at a point z_0 and $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{1}$$

It can be extended to the case where the first $m-1$ derivatives of both f and g are zero by using Taylor series. That is

Theorem 1. *If f and g are both analytic at z_0 and*

$$f(z_0) = g(z_0) = f'(z_0) = g'(z_0) = \ldots = f^{(m-1)}(z_0) = g^{(m-1)}(z_0) = 0 \tag{2}$$

then if $g^{(m)}(z_0) \neq 0$

$$\lim_{z \to z_0} \frac{f(z_0)}{g(z_0)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \tag{3}$$

Proof. Because of 2, all the terms up to and including the $(z-z_0)^{m-1}$ term in the Taylor series for both f and g are zero, so

$$\begin{aligned} f(z_0) &= \sum_{k=m}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \\ g(z_0) &= \sum_{k=m}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k \end{aligned} \tag{4}$$

Dividing the f series by the g series we have

$$\lim_{z \to z_0} \frac{f(z_0)}{g(z_0)} = \lim_{z \to z_0} \frac{\sum_{k=m}^{\infty} f^{(k)}(z_0)(z-z_0)^k / k!}{\sum_{k=m}^{\infty} g^{(k)}(z_0)(z-z_0)^k / k!} \tag{5}$$

Dividing top and bottom by $(z-z_0)^m$ we have

$$\lim_{z\to z_0} \frac{\sum_{k=m}^{\infty} f^{(k)}(z_0)(z-z_0)^k}{\sum_{k=m}^{\infty} g^{(k)}(z_0)(z-z_0)^k} = \lim_{z\to z_0} \frac{\sum_{k=m}^{\infty} f^{(k)}(z_0)(z-z_0)^{k-m}/k!}{\sum_{k=m}^{\infty} g^{(k)}(z_0)(z-z_0)^{k-m}/k!} \tag{6}$$

$$= \lim_{z\to z_0} \frac{f^{(m)}(z_0)/m! + \sum_{k=1}^{\infty} f^{(m+k)}(z_0)(z-z_0)^k/(m+k)!}{g^{(m)}(z_0)/m! + \sum_{k=1}^{\infty} g^{(m+k)}(z_0)(z-z_0)^k/(m+k)!} \tag{7}$$

$$= \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \tag{8}$$

This follows because both sums in 7 go to zero in the limit because of the $(z-z_0)^k$ terms. □