GENERALIZED PRODUCT RULE - LEIBNIZ'S FORMULA

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The product rule for derivatives is, for two functions $f(x)$ and $g(x)$:

$$(fg)' = f'g + fg' \tag{1}$$

This rule can be generalized to give Leibniz's formula for the n^{th} derivative of a product:

$$(fg)^{(n)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) f^{(k)} g^{(n-k)} \tag{2}$$

where the notation $f^{(n)} \equiv d^n f/dx^n$ and $\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

The proof of this formula is a nice little exercise in proof by mathematical induction. An inductive proof requires two steps. First, we must show that the formula is true for one particular value of n, say $n = 0$. Second, we *assume* that the formula is true for a value $n = m$. From this assumption we then must prove that the formula is also true for $n = m + 1$. It doesn't matter what m is as long as it is taken to be equal to or greater than the value of n used in step 1. The idea is that if we can show that the truth of the formula for a particular value of n always implies its truth for the next value of n, then as long as we can demonstrate the truth of the formula for *some* value of n (as we do in step 1), the inductive reasoning implies it is true for all n greater than or equal to the specific value.

The first step in the proof is called the *anchor step* and the second step the *inductive step*.

In our case, taking $n = 0$ gives us the identity $(fg)^{(0)} = fg$ which is clearly true. So now we prove the inductive step. Assuming the formula is true for $n = m$ gives us

$$(fg)^{(m)} = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array}\right) f^{(k)} g^{(m-k)} \tag{3}$$

Taking the derivative of this, and applying the regular product rule, gives

$$(fg)^{(m+1)} = \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \left(f^{(k+1)} g^{(m-k)} + f^{(k)} g^{(m-k+1)}\right) \tag{4}$$

In the first term, we can shift the summation index by replacing k by $k-1$ to get

$$(fg)^{(m+1)} = \sum_{k=1}^{m+1} \begin{pmatrix} m \\ k-1 \end{pmatrix} f^{(k)} g^{(m-k+1)} + \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} f^{(k)} g^{(m-k+1)} \tag{5}$$

Note the limits on the two sums. We can now combine the sums in the regions where their indexes overlap to get

$$(fg)^{(m+1)} = fg^{(m+1)} + \sum_{k=1}^{m} \left[\begin{pmatrix} m \\ k-1 \end{pmatrix} + \begin{pmatrix} m \\ k \end{pmatrix} \right] f^{(k)} g^{(m-k+1)} + f^{(m+1)} g \tag{6}$$

We can work out the sum of the two binomial coefficients by expanding them in terms of factorials:

$$\begin{pmatrix} m \\ k-1 \end{pmatrix} + \begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{(k-1)!(m-k+1)!} + \frac{m!}{k!(m-k)!} \tag{7}$$

Putting over a common denominator:

$$= \frac{(k)m!}{k!(m-k+1)!} + \frac{(m-k+1)m!}{k!(m-k+1)!} \tag{8}$$

$$= \frac{(m+1)!}{k!(m-k+1)!} \tag{9}$$

$$= \begin{pmatrix} m+1 \\ k \end{pmatrix} \tag{10}$$

Since $\begin{pmatrix} m+1 \\ 0 \end{pmatrix} = \begin{pmatrix} m+1 \\ m+1 \end{pmatrix} = 1$ we can combine the two outlying terms in 6 to get the final result

$$(fg)^{(m+1)} = \sum_{k=0}^{m+1} \begin{pmatrix} m+1 \\ k \end{pmatrix} f^{(k)} g^{(m-k+1)} \tag{11}$$

This completes the inductive proof and establishes Leibniz's formula.

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