GREEN'S FUNCTIONS; FORCED HARMONIC OSCILLATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

To introduce the idea of Green's functions, let's return to the classical harmonic oscillator. Its equation of motion is

$$m\ddot{x}+kx=0 \tag{1}$$

This has the general solution

$$x(t)=A\sin\omega t+B\cos\omega t \tag{2}$$

where

$$\omega=\sqrt{\frac{k}{m}} \tag{3}$$

and A and B are constants determined by the initial conditions.

Suppose now we add a forcing term to the oscillator. That is, in addition to the oscillator's innate restoring force term kx, we drive the oscillator with an additional force $F(t)$. The equation of motion is now

$$m\ddot{x}+kx=F(t) \tag{4}$$

Since this driving force can be anything, we obviously cannot solve this equation until we know $F(t)$.

Now suppose we had a function $G(t-t')$ which satisfied the equation

$$m\frac{\partial^2 G(t,t')}{\partial t^2}+kG\left(t,t'\right)=\delta\left(t-t'\right) \tag{5}$$

where δ is Dirac delta function. Then it turns out that the solution to 4 is

$$x(t)=\int_{-\infty}^{\infty}G\left(t,t'\right)F\left(t'\right)dt' \tag{6}$$

This can be verified by direct substitution:

$$
\begin{aligned}
m\ddot{x}+kx &= \int_{-\infty}^{\infty} m\frac{\partial^{2}G\left(t,t'\right)}{\partial t^{2}}F\left(t'\right)dt'+\int_{-\infty}^{\infty}kG\left(t,t'\right)F\left(t'\right)dt' && (7)\\
&= \int_{-\infty}^{\infty}\left[m\frac{\partial^{2}G\left(t,t'\right)}{\partial t^{2}}+kG\left(t,t'\right)\right]F\left(t'\right)dt' && (8)\\
&= \int_{-\infty}^{\infty}\delta\left(t-t'\right)F\left(t'\right)dt' && (9)\\
&= F\left(t\right) && (10)
\end{aligned}
$$

The function $G\left(t,t'\right)$ is called the *Green's function*. If we know G, we can convert the original differential equation 4 into an ordinary integral 6 which is presumably easier to do.

The catch, of course, is that finding the Green's function is seldom easy and can often turn out to be just as difficult as solving the differential equation directly. For the harmonic oscillator, however, finding the Green's function is possible, so here we go.

To make things definite, we'll take $F\left(t\right)=\delta\left(t\right)$ in 4. This corresponds to an impulse force applied to the oscillator at $t=0$. We'll also assume that prior to $t=0$, the oscillator was at rest, so $x=0$ for $t<0$. The forced oscillator equation of motion is thus

$$m\ddot{x}+kx=\delta\left(t\right) \tag{11}$$

For $t>0$, $F\left(t\right)=0$, so the solution in this region must reduce to the unforced oscillator 2. To determine A and B, we need to impose boundary conditions at $t=0$, so we'll need to investigate what happens here in a bit more detail. The process is reminiscent of the calculations we did when analyzing the delta function potential in quantum mechanics.

Since all the action occurs around $t=0$, we'll integrate 11 over a small interval $t\in\left[-\varepsilon,\varepsilon\right]$:

$$\int_{-\varepsilon}^{\varepsilon}m\ddot{x}\left(t\right)dt+\int_{-\varepsilon}^{\varepsilon}kx\left(t\right)dt=\int_{-\varepsilon}^{\varepsilon}\delta\left(t\right)dt \tag{12}$$

Term by term, we get

$$
\begin{aligned}
\int_{-\varepsilon}^{\varepsilon}m\ddot{x}\left(t\right)dt &= m\,\dot{x}\left(t\right)\Big|_{-\varepsilon}^{\varepsilon} && (13)\\
\left|\int_{-\varepsilon}^{\varepsilon}kx\left(t\right)dt\right| &< 2\varepsilon k\max\left|x\left(t\right)\right| && (14)\\
\int_{-\varepsilon}^{\varepsilon}\delta\left(t\right)dt &= 1 && (15)
\end{aligned}
$$

In the limit $\varepsilon \to 0$, the second line tends to zero, since we're assuming that the maximum displacement of the oscillator is finite. Therefore, we must have

$$\lim_{\varepsilon\to 0} m\,\dot{x}(t)\big|_{-\varepsilon}^{\varepsilon} = m\dot{x}_{0^+} - m\dot{x}_{0^-} = 1 \tag{16}$$

That is, the velocity of the oscillator is discontinuous at $t=0$. Since the oscillator is at rest for $t<0$, $\dot{x}_{0^-}=0$ so

$$\dot{x}_{0^+} = \frac{1}{m} \tag{17}$$

[By the way, a discontinuous velocity implies an infinite acceleration, but that's what we would expect given that the force applied at $t=0$ is a delta function, which is infinite.]

The displacement x is continuous across $t=0$, since the velocity is finite and in the limit $\varepsilon \to 0$

$$\left|\int_{-\varepsilon}^{\varepsilon} \dot{x}(t)\,dt\right| < 2\varepsilon \max|\dot{x}| \to 0 \tag{18}$$

Therefore, since $x=0$ for $t<0$ and $x = A\sin\omega t + B\cos\omega t$ for $t>0$, we must have $x(0)=0$ so $B=0$. From 17 we have

$$\dot{x}\left(0^+\right) = \frac{1}{m} = A\omega \tag{19}$$

$$A = \frac{1}{m\omega} \tag{20}$$

Thus the general solution to the forced oscillator with a delta function forcing term is

$$x(t) = \begin{cases} 0 & t<0 \\ \frac{1}{m\omega}\sin\omega t & t>0 \end{cases} \tag{21}$$

If we shift the origin from $t=0$ to $t=t'$, the equation of motion 11 becomes

$$m\ddot{x} + kx = \delta\left(t-t'\right) \tag{22}$$

and the solution becomes

$$x\left(t,t'\right) = \begin{cases} 0 & t<t' \\ \frac{1}{m\omega}\sin\omega\left(t-t'\right) & t>t' \end{cases} \tag{23}$$

Note that 22 has exactly the same form as the Green's function equation 5, so we have actually found the Green's function for the harmonic oscillator. This means that a general solution to the forced oscillator is given by

$$x(t) = \int_{-\infty}^{\infty} x\left(t,t'\right) F\left(t'\right) dt' \tag{24}$$

$$= \frac{1}{m\omega} \int_{-\infty}^{t} \sin\omega\left(t-t'\right) F\left(t'\right) dt' \tag{25}$$

Note that the upper limit of the integral becomes t in the last line, since $x(t,t') = 0$ for $t' > t$.

A physical interpretation of this solution is that we're adding up the contributions from the forcing function that occurred before the specified time t. Clearly what the forcing function $F(t')$ does in the future (that is, for $t' > t$) can't affect the position $x(t)$ at the current time t, but all values of $F(t')$ in the past *do* affect the oscillator's current position.

A Green's function that is zero before a certain point, as our oscillator Green's function is, is called a *retarded Green's function*.

PINGBACKS

Pingback: Green's function for Klein-Gordon equation
Pingback: Klein-Gordon Green's function as a Fourier transform