GROUP THEORY: DEFINITIONS & EXAMPLES, INCLUDING THE EUCLIDEAN GROUP

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Since group theory figures prominently in quantum field theory and particle physics, and since I never took a course that covered it (this was back in the 1970s when I was doing a physics and astronomy undergrad degree; things may have changed now), I thought it would be useful to do a few posts covering the basics. I'm not following any textbook for this (although I'm sure there are many excellent books out there); rather, I'm gleaning most of the information from scattered web sites. A good site that starts from an elementary level is the Dog School of Mathematics, but you may find a site more to your taste via Google.

First, we need to introduce some definitions and axioms. A *group* consists of a set of objects (they are usually mathematical objects, but in principle, you could use any objects in everyday life, if they satisfy the following conditions) and an *operation* that can be performed on these objects. Notation varies, but if we have two objects A and B, we denote the operation by a dot $\bullet$. (To avoid confusion, this symbol does not necessarily represent multiplication, although it *could*.)

The set of objects is said to form a group under that particular operation if it satisfies the following conditions:

(1) **Completeness.** If $A\bullet B=C$, then C must also be in the group. For example, the set of all integers is complete under the operation of multiplication (the product of any two integers gives another integer) and also under the operation of addition (the sum of any two integers is always another integer). The set of *positive* integers is also complete under both multiplication and addition, but the set of *negative* integers is *not* complete under multiplication, since the product of any two negative integers is a positive integer. The set of negative integers *is* complete under addition, however.

(2) **Associativity.** This states that $A\bullet(B\bullet C)=(A\bullet B)\bullet C$. That is, in a sequence of two operations on three objects, it doesn't matter if we perform the left or right operation first. The integers are associative under multiplication and addition, but not under subtraction. For example, $(10-5)-2=3\neq10-(5-2)=7$.

(3) **Identity element.** A group must contain an element E called the *identity* which has the property that combining it with any element of the group leaves that element unchanged. That is, $E \bullet A = A \bullet E = A$. For integers under multiplication, $E = 1$ (multiplying by 1 leaves any integer unchanged). Under addition or subtraction, $E = 0$ (adding or subtracting 0 leaves the integer unchanged).

(4) **Inverse.** Every element A in a group must have an inverse A^{-1} with the property $A \bullet A^{-1} = A^{-1} \bullet A = E$. Again looking at integers, we see that an inverse exists for addition ($A^{-1} = -A$) since $A + (-A) = 0 = E$. If the set contains only integers, there is no inverse for multiplication (unless $A = \pm 1$, which is its own inverse). Clearly if the set doesn't have an identity element, it can't have an inverse either.

A few examples:

- The set of all integers is a group under addition and also under subtraction. It is *not* a group under multiplication (or division), since not every element has an inverse (in fact, only 1 and -1 have inverses that are integers).
- The set of all rational numbers is also a group under addition and also under subtraction. However, it is still *not* a group under multiplication, since 0 does not have an inverse. The set of all rational numbers *excluding* 0, however, *is* a group under multiplication and division.
- The set of vectors in n-dimensional space is a group under vector addition. The sum of two vectors always produces another vector (completeness), the order in which 3 vectors are added always gives the same result (associativity), adding a zero vector to another vector leaves the second vector unchanged (identity) and adding the negative of a vector to itself gives the zero vector (inverse).
- The set of vectors is *not* a group under either the scalar (dot) or vector (cross) product, however. The scalar product of two vectors produces a scalar, which is not a vector and therefore not in the set, so the scalar product is not complete. The cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ always produces either the zero vector (if $\mathbf{A} \perp \mathbf{B}$) or another vector $\mathbf{C}$ that is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Thus there is no identity vector $\mathbf{E}$ such that $\mathbf{A} \times \mathbf{E} = \mathbf{A}$ for all vectors $\mathbf{A}$. Because there is no identity vector, there can also be no inverse to the cross product. [It *is* true that we can find *three* vectors such $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ and $\mathbf{B} \times \mathbf{C} = \mathbf{A}$ so we can get $\mathbf{A}$ back again after two products (for example, the three unit vectors along the coordinate axes), but we need two vectors besides $\mathbf{A}$ to do this, while the identity condition

says there has to be a *single* vector $\mathbf{E}$ such that $\mathbf{A}\times\mathbf{E}=\mathbf{A}$, and that's impossible.]

Euclidean groups. A Euclidean group is the set of transformations of points in a plane that leave all distances between points unchanged. This group contains rigid rotations about some fixed point and rigid translations. Besides its importance in mathematics, this group also lies at the foundation of computer graphics, since it forms the basics of transformations of shapes in a display.

Each element of the set is itself an operation that is performed on all the points (x,y) in the plane. We can represent a rotation by $R(x_{r,},y_r,\theta)$ where (x_r,y_r) is the axis of rotation and θ is the angle. Similarly, a translation can be written as $T(\Delta x,\Delta y)$ where $(\Delta x,\Delta y)$ is the displacement. The operator $\bullet$ applies if we wish to perform two or more transformations. Thus to perform a rotation followed by a translation we would write

$$T(\Delta x,\Delta y)\bullet R(x_{r,},y_r,\theta) \tag{1}$$

It's usual in this notation for the order of operations to be read from right to left, so in this case the rotation is done first, then the translation. The operator $\bullet$ can therefore be read as 'followed by' if we read the line in that order.

To demonstrate that this set of transformations is indeed a group, we look at the four conditions:

(1) Completeness: Fairly obviously, the set of translations (without rotations) is complete, since if we translate the plane by an amount $(\Delta x_1,\Delta y_1)$ and then by $(\Delta x_2,\Delta y_2)$, the result is just another translation $(\Delta x_1+\Delta x_2,\Delta y_1+\Delta y_2)$. To bring rotations into the picture, we observe that to perform a rotation by some angle θ about some arbitrary point (x_r,y_r), we can first translate the plane so that the axis of rotation is at the origin using $T(-x_r,-y_r)$, do the rotation through θ using $R(0,0,\theta)$, then translate the plane back to (x_r,y_r) using $T(x_r,y_r)$. Thus any rotation is a combination of two translations and a rotation about the origin: $R(x_{r,},y_r,\theta)=T(x_r,y_r)\bullet R(0,0,\theta)\bullet T(-x_r,-y_r)$, so the result of any rotation produces a transformation that is part of the set.

(2) Associativity. This is actually fairly trivial, since the transformations in a sequence are always performed from right to left, and we can't apply a transformation until we have the result of the one preceding it. That is, if we apply three transformations in the order $T_1\bullet R_2\bullet R_1$, say, it doesn't really matter whether we put parentheses in this expression, since we can't do R_2 until we see the result of

R_1, and we can't do T_1 until we have the result of $(R_2 \bullet R_1)$. That is, $(T_1 \bullet R_2) \bullet R_1 = T_1 \bullet (R_2 \bullet R_1)$.

(3) The identity transformation is just doing nothing at all. That is, a translation where $\Delta x = \Delta y = 0$ or a rotation where $\theta = 0$.

(4) The inverse transformation is just the transformation that undoes the original. For a translation $T^{-1}(\Delta x, \Delta y) = T(-\Delta x, -\Delta y)$ and for a rotation $R^{-1}(x_r, y_r \theta) = R(x_r, y_r - \theta)$. The inverse of a sequence of transformations is a sequence of inverses, applied in the reverse order. That is, for example, $[T_1 \bullet R_2 \bullet R_1]^{-1} = R_1^{-1} \bullet R_2^{-1} \bullet T_1^{-1}$.

One point worthy of note here is that the definition of a group does *not* require the operation to be commutative. That is, it is *not* required that $A \bullet B = B \bullet A$. For our earlier examples involving integers and vectors, the groups actually *were* commutative, but the Euclidean group is not. You can see this by trying a translation followed by a rotation about the origin. If you reverse the order (do the rotation first, then the translation), you don't get the same result. For example, suppose we translate by $T(1,-1)$ and then rotate by $R\left(0,0,\frac{\pi}{2}\right)$. The point initially at $(1,0)$ first gets shifted by the translation to $(2,-1)$. The rotation then moves this point to $(1,2)$.

Now if we go back to the point $(1,0)$ and then rotate it first, we get the point $(0,1)$. The translation then gives the point $(1,0)$, giving the original point back again. Since $(1,2) \neq (1,0)$, the two operations do not commute.

Groups that *do* commute are known as *Abelian groups*. [The only group theory joke I know is: "What's purple and commutes?" "The Abelian grape." But I digress...]

PINGBACKS

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