HARMONIC CONJUGATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 8 December 2024.

The real and imaginary components $u(x,y)$ and $v(x,y)$ of an analytic complex function $f(z)$ are both harmonic functions, in that they satisfy Laplace's equation

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \\ v_{xx}+v_{yy} &= 0 \end{aligned} \tag{1}$$

where I've used the shorthand notation of a subscript indicating a derivative, so that $u_x \equiv \frac{\partial u}{\partial x}$ and so on.

If we're given the function $u(x,y)$, we can use the Cauchy-Riemann equations (CR) to find a suitable candidate for $v(x,y)$. The functions u and v are known as *harmonic conjugates* of each other. This means that they are both harmonic functions, and that they satisfy CR. Before giving a few examples, we'll give a few theorems concerning harmonic conjugates.

Theorem 1. *If $v(x,y)$ is a harmonic conjugate of $u(x,y)$ in a domain D, then* every *harmonic conjugate of u has the form $v+a$ where a is a real constant.*

Proof. Since u and v are components of an analytic function, they satisfy CR. This means that

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{2}$$

Let's assume that there is another function $w(x,y)=v(x,y)+\psi(x,y)$ that is a harmonic conjugate of u. Then w must also satisfy CR, so

$$\begin{aligned} u_x &= v_y+\psi_y \\ u_y &= -v_x-\psi_x \end{aligned} \tag{3}$$

However, from 2 we have

$$\begin{aligned} v_y+\psi_y &= v_y \\ -v_x-\psi_x &= -v_x \end{aligned} \tag{4}$$

Therefore

$$\psi_y = 0 \tag{5}$$

which means that ψ can be a function of x alone. However, we also have

$$\psi_x = 0 \tag{6}$$

so that ψ can't depend on either x or y, so must be a constant. Thus the most general harmonic conjugate is $v(x,y) + a$. Since both u and its conjugate must be real functions, a must be a real constant. □

Theorem 2. *If v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v.*

Proof. We are given that u and v satisfy CR. If we swap u and v and change the sign of u so that $-u$ becomes the imaginary component and v the real component, then the new form of CR is

$$\begin{aligned} v_x &= -u_y \\ v_y &= -(-u_x) = u_x \end{aligned} \tag{7}$$

However, these are just the original CR conditions. □

Theorem 3. *If v is a harmonic conjugate for u, then the product uv is a harmonic function.*

Proof. We can verify this by direct differentiation. We have

$$[uv]_{xx} + [uv]_{yy} = [u_x v + u v_x]_x + [u_y v + u v_y]_y \tag{8}$$

$$= u_{xx} v + u_x v_x + u_x v_x + u v_{xx} + u_{yy} v + u_y v_y + u_y v_y + u v_{yy} \tag{9}$$

$$= (u_{xx} + u_{yy}) v + 2(u_x v_x + u_y v_y) + (v_{xx} + v_{yy}) u \tag{10}$$

Because u and v are harmonic, the first and last terms are zero. Using CR we can modify the second term using $v_x = -u_y$ and $v_y = u_x$, so we have

$$u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0 \tag{11}$$

Thus $[uv]_{xx} + [uv]_{yy} = 0$ and is harmonic. □

We now present a few examples of finding harmonic conjugates. In each case, we're given u and must find v.

Example 1. $u = y$. From CR

$$u_x = 0 = v_y \tag{12}$$

Thus

$$v = b + \psi(x) \tag{13}$$

where b is a constant. Also from CR

$$u_y = 1 = -v_x = -\psi_x \tag{14}$$

so

$$\psi(x) = -x + c \tag{15}$$

where c is another constant. We can merge b and c into a single constant a and we have

$$v(x,y) = -x + a \tag{16}$$

Example 2. $u = e^x \sin y$. From CR

$$u_x = e^x \sin y = v_y \tag{17}$$

$$v = -e^x \cos y + \psi(x) \tag{18}$$

$$u_y = e^x \cos y = -v_x = e^x \cos y - \psi_x \tag{19}$$

Thus $\psi_x = 0$ and

$$v(x,y) = -e^x \cos y + a \tag{20}$$

Example 3. $u = xy - x + y$. From CR

$$u_x = y - 1 = v_y \tag{21}$$

$$v = \frac{1}{2}y^2 - y + \psi(x) \tag{22}$$

$$u_y = x + 1 = -v_x = -\psi_x \tag{23}$$

$$\psi(x) = -\frac{1}{2}x^2 - x + a \tag{24}$$

so the conjugate is

$$v(x,y) = \frac{1}{2}\left(y^2 - x^2\right) - x - y + a \tag{25}$$

Example 4. $u = \sin x \cosh y$. From CR

$$u_x = \cos x \cosh y = v_y \tag{26}$$
$$v = \cos x \sinh y + \psi(x) \tag{27}$$
$$u_y = \sin x \sinh y = -v_x = \sin x \sinh y - \psi_x \tag{28}$$

Thus $\psi_x = 0$ and the conjugate is

$$v(x,y) = \cos x \sinh y + a \tag{29}$$

Example 5. $u = \ln|z|$ for $\Re z > 0$. This one gets a bit messier, so I used Maple to work out the derivatives. We have

$$u = \ln\left(\sqrt{x^2+y^2}\right) \tag{30}$$

so

$$u_x = \frac{x}{x^2+y^2} = v_y \tag{31}$$
$$v = \arctan\frac{y}{x} + \psi(x) \tag{32}$$
$$u_y = \frac{y}{x^2+y^2} = -v_x \tag{33}$$
$$v = -\int u_y dx = -\arctan\frac{x}{y} \tag{34}$$

We can use the trig identity

$$\arctan\frac{y}{x} = \frac{\pi}{2} - \arctan\frac{x}{y} \tag{35}$$

so we have

$$v(x,y) = \arctan\frac{y}{x} - \frac{\pi}{2} \tag{36}$$

Actually, the $-\frac{\pi}{2}$ could be any constant, since it is only derivatives that figure in CR and Laplace's equation. Thus

$$v(x,y) = \arctan\frac{y}{x} + a \tag{37}$$

Example 6. $u = \Im e^{z^2}$. Again, we turn to Maple for help in calculating the derivatives and algebra. We have

$$u = e^{x^2-y^2}\sin(2xy) \tag{38}$$

so

$$u_x = 2xe^{x^2-y^2}\sin(2xy) + 2e^{x^2-y^2}y\cos(2xy) = v_y \tag{39}$$

$$v = \frac{e^{x^2-y^2}\left(\tan^2(xy)-1\right)}{1+\tan^2(xy)} + \psi(x) \tag{40}$$

$$= \left(-2\cos^2(xy)+1\right)e^{x^2-y^2} + \psi(x) \tag{41}$$

$$= -\cos(2xy)\,e^{x^2-y^2} + \psi(x) \tag{42}$$

The second CR equation gives us

$$u_y = -2ye^{x^2-y^2}\sin(2xy) + 2e^{x^2-y^2}x\cos(2xy) = -v_x \tag{43}$$

$$v = \frac{e^{x^2-y^2}\left(\tan^2(xy)-1\right)}{1+\tan^2(xy)} + a \tag{44}$$

$$= -\cos(2xy)\,e^{x^2-y^2} + a \tag{45}$$

Comparing this with 41 we see that

$$v(x,y) = -\cos(2xy)\,e^{x^2-y^2} + a \tag{46}$$

In case you're curious, I did actually verify Laplace's equation for u and v as given by 38 and 46 using Maple.

There's actually a quicker way to do this example, making use of Theorem 2 above. We have

$$e^{z^2} = e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right) \tag{47}$$

Since this is an analytic function, its imaginary part is a harmonic conjugate for its real part, so the negative of the real part is the harmonic conjugate for the imaginary part. That is, the harmonic conjugate of $\Im e^{z^2}$ is $-\Re e^{z^2}$ which is

$$-\Re e^{z^2} = -\cos(2xy)\,e^{x^2-y^2} \tag{48}$$

Adding a real constant a to get the general form gives 46.