# HARMONIC FUNCTIONS

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A real function $\phi(x,y)$ of two variables is called a *harmonic function* if it satisfies Laplace's equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{1}$$

**Theorem 1.** *It's a remarkable fact that the real and imaginary parts of any analytic function $f(z)=u(x,y)+iv(x,y)$ are both harmonic functions.*

*Proof.* To see this, we use the Cauchy-Riemann equations (CR). In what follows, I use a subscript $x$ to indicate a partial derivative with respect to $x$, and similarly for $y$. We have the CR equations:

$$u_x = v_y \tag{2}$$
$$u_y = -v_x \tag{3}$$

Therefore

$$u_{xx} = v_{yx} \tag{4}$$
$$u_{yy} = -v_{xy} \tag{5}$$

Since the order of differentiation in a second partial derivative commutes, we have

$$u_{xx}+u_{yy}=v_{xy}-v_{xy}=0 \tag{6}$$

A similar calculation shows that

$$v_{xx}+v_{yy}=0 \tag{7}$$

Thus the harmonic property of $u$ and $v$ follows from the requirement that the derivative of an analytic complex function must be the same, no matter in which direction we approach the limit point. □

Some examples of harmonic functions follow. In each case, the function is considered over its domain of definition, where it is analytic.

**Example 1.** The function $f(z) = z^2 + 2z + 1$ is analytic everywhere. To verify 6 and 7 we calculate $u$ and $v$. I've used Maple to do some of the algebra.

$$u(x,y) = x^2 - y^2 + 2x + 1 \tag{8}$$

$$v(x,y) = 2xy + 2y \tag{9}$$

Calculating derivatives, we have

$$u_{xx} = 2 \tag{10}$$

$$u_{yy} = -2 \tag{11}$$

$$u_{xx} + u_{yy} = 0 \tag{12}$$

and

$$v_{xx} = 0 = v_{yy} \tag{13}$$

$$v_{xx} + v_{yy} = 0 \tag{14}$$

**Example 2.** For the function $g(z) = \frac{1}{z}$ we have

$$u(x,y) = \frac{x}{x^2 + y^2} \tag{15}$$

$$v(x,y) = \frac{-y}{x^2 + y^2} \tag{16}$$

Calculating the derivatives is a bit tedious, so I used Maple to get

$$u_{xx} = \frac{8x^3}{(x^2 + y^2)^3} - \frac{6x}{(x^2 + y^2)^2} \tag{17}$$

$$= \frac{2x\left(x^2 - 3y^2\right)}{(x^2 + y^2)^3} \tag{18}$$

$$u_{yy} = \frac{8xy^2}{(x^2 + y^2)^3} - \frac{2x}{(x^2 + y^2)^2} \tag{19}$$

$$= -\frac{2x\left(x^2 - 3y^2\right)}{(x^2 + y^2)^3} \tag{20}$$

$$= -u_{xx} \tag{21}$$

and for $v$

$$v_{xx} = \frac{8yx^2}{(x^2+y^2)^3} + \frac{2y}{(x^2+y^2)^2} \tag{22}$$

$$= \frac{-6x^2y+2y^3}{(x^2+y^2)^3} \tag{23}$$

$$v_{yy} = \frac{8y^3}{(x^2+y^2)^3} + \frac{6y}{(x^2+y^2)^2} \tag{24}$$

$$= \frac{6x^2y-2y^3}{(x^2+y^2)^3} \tag{25}$$

$$= -v_{xx} \tag{26}$$

Thus both $u$ and $v$ are harmonic.

**Example 3.** For $h(z) = e^z$ we have

$$u(x,y) = e^x \cos y \tag{27}$$

$$v(x,y) = e^x \sin y \tag{28}$$

The derivatives are

$$u_{xx} = e^x \cos y \tag{29}$$

$$u_{yy} = -e^x \cos y \tag{30}$$

$$v_{xx} = e^x \sin y \tag{31}$$

$$v_{yy} = -e^x \sin y \tag{32}$$

Thus $e^z$ is harmonic.

**Example 4.** Find the most general harmonic polynomial of form

$$\phi(x,y) = ax^2 + bxy + cy^2 \tag{33}$$

where $a, b, c$ are constants.

This is a single real function, so calculating derivatives we have

$$\phi_{xx} = 2a \tag{34}$$

$$\phi_{yy} = 2c \tag{35}$$

Therefore we have $\phi_{xx} + \phi_{yy} = 0$ if $a = -c$. The constant $b$ can be anything.

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