# HARNACK'S INEQUALITY

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A consequence of the Poisson integral formula is Harnack's inequality, which can be stated as the theorem:

**Theorem 1.** *If $\phi(z)$ is harmonic and nonnegative in a domain containing the disk $|z| \le R$ then for $0 \le r < R$*

$$\phi(0)\frac{R-r}{R+r} \le \phi\left(re^{i\theta}\right) \le \phi(0)\frac{R+r}{R-r} \tag{1}$$

*Proof.* First, we recall the Poisson integral formula:

$$\phi\left(re^{i\theta}\right) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi\left(Re^{it}\right)}{R^2+r^2-2rR\cos(t-\theta)}dt \tag{2}$$

We also need a couple of other inequalities. Observe that

$$(R-r)^2 = R^2+r^2-2rR \le R^2+r^2-2rR\cos(t-\theta) \tag{3}$$

This follows because $\cos(t-\theta)$ lies in the interval $[-1,1]$ so the $-2rR$ term on the LHS is always less than or equal to $-2rR\cos(t-\theta)$.

For the same reason

$$(R+r)^2 = R^2+r^2+2rR \ge R^2+r^2-2rR\cos(t-\theta) \tag{4}$$

Now rewrite the first and last terms in 1

$$\phi(0)\frac{R-r}{R+r}\times\frac{R+r}{R+r} = \phi(0)\frac{R^2-r^2}{(R+r)^2} \tag{5}$$

$$\phi(0)\frac{R+r}{R-r}\times\frac{R-r}{R-r} = \phi(0)\frac{R^2-r^2}{(R-r)^2} \tag{6}$$

From 2 we have, using 4

$$\phi(0)\frac{R-r}{R+r} = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi\left(Re^{it}\right)}{(R+r)^2}dt \tag{7}$$

$$\leq \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi\left(Re^{it}\right)}{R^2+r^2-2rR\cos(t-\theta)}dt = \phi\left(re^{i\theta}\right) \tag{8}$$

This proves the left-hand inequality in 1. For the right-hand inequality we have, using 3

$$\phi(0)\frac{R+r}{R-r} = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi\left(Re^{it}\right)}{(R-r)^2}dt \tag{9}$$

$$\geq \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi\left(Re^{it}\right)}{R^2+r^2-2rR\cos(t-\theta)}dt = \phi\left(re^{i\theta}\right) \tag{10}$$

This proves the right-hand inequality. Note that these inequalities work only if $\phi\left(re^{i\theta}\right)$ is nonnegative, since if it's negative, the inequalities would be opposite in direction. □

**Theorem 2.** *Liouville's theorem for harmonic functions. If $\phi$ is harmonic in the whole plane and bounded from above or below, then $\phi$ is constant.*

*Proof.* From 1 we see that $\phi\left(re^{i\theta}\right)$ is squeezed between two limits. Since $\phi$ is bounded, then $\phi(0)$ is finite, so 1 must be true for any $R$. In particular, we take the limit as $R \to \infty$ to get

$$\lim_{R\to\infty}\phi(0)\frac{R-r}{R+r} = \phi(0) \leq \phi\left(re^{i\theta}\right) \tag{11}$$

$$\lim_{R\to\infty}\phi(0)\frac{R+r}{R-r} = \phi(0) \geq \phi\left(re^{i\theta}\right) \tag{12}$$

Thus we must have

$$\phi\left(re^{i\theta}\right) = \phi(0) \tag{13}$$

over the entire plane. □