HERMITE DIFFERENTIAL EQUATION - GENERATING FUNCTIONS

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Hermite's differential equation shows up during the solution of the Schrödinger equation for the harmonic oscillator. The differential equation can be written in the form

$$\frac{d^2f}{dy^2} - 2y\frac{df}{dy} + (\epsilon - 1)f = 0 \tag{1}$$

but an analysis of the series solution of this equation shows that the parameter ϵ has to have the form

$$\epsilon = 2n + 1 \tag{2}$$

for some integer n, so we can rewrite the differential equation as

$$\frac{d^2f}{dy^2} - 2y\frac{df}{dy} + 2nf = 0 \tag{3}$$

We know the solutions of this equation are polynomials in y, and we got (from the series solution) a recursion formula for the coefficients of the polynomials, but a recursion formula can be difficult to work with, and it turns out that there is another form that can be used to work with these polynomials. This uses the idea of the *generating function*.

The idea is that we can write a function $S(y,s)$, where y is the same y as in the differential equation, and s is a kind of dummy variable that allows us to do calculations (as we'll see in a moment). Suppose we define this function as follows:

$$S(y,s) \equiv e^{-s^2+2sy} \tag{4}$$

From the expansion of the exponential in a Taylor series, we can also write this as

$$
\begin{aligned}
S(y,s) &= \sum_{m=0}^{\infty}\frac{(-s^2+2sy)^m}{m!} &(5)\\
&= \sum_{m=0}^{\infty}\frac{s^m(2y-s)^m}{m!} &(6)
\end{aligned}
$$

At first (and probably second) glance, this formula seems to have little relation to Hermite polynomials, but let's write out the first few terms of the series

$$
\begin{aligned}
S(y,s) &= 1+\frac{s(2y-s)}{1!}+\frac{s^2(2y-s)^2}{2!}+\frac{s^3(2y-s)^3}{3!}+\ldots &(7)\\
&= 1+2ys+(-1+2y^2)s^2+(-2y+\frac{4}{3}y^3)s^3+\ldots &(8)
\end{aligned}
$$

In the second line, we regrouped the series so that terms with the same power of s are grouped together. The m^{th} term in the series contains terms involving s to the m^{th} and higher powers only, so if we want to isolate those terms for a particular power (say the n^{th} power) of s we need look at only the first n terms of the series. What do we get if we look at terms involving each successive power of s, starting with the zeroth power? As can be seen above, the term involving s^m is multiplied by a polynomial in y and by comparing these polynomials with those obtained by our earlier definition of the Hermite polynomials, we can see that each polynomial here is $H_m(y)/m!$. That is

$$S(y,s)=\sum_{m=0}^{\infty}\frac{H_m(y)}{m!}s^m \tag{9}$$

Obviously we haven't *proved* this in general, but this function may also be taken as the definition of Hermite polynomials, as the other definition that we used earlier can be derived from it, as we'll see at the end of this post.

The Hermite polynomials can be obtained from this generating function by taking derivatives, as follows. Since the j^{th} derivative of s^m is zero if $m<j$, taking this derivative will eliminate all terms with $m<j$. The j^{th} derivative of s^j is the constant $j!$. For all higher powers where $m>j$, the j^{th} derivative will leave a term s^{m-j}. So if we take the j^{th} derivative of $S(y,s)$ and then set $s=0$ we will isolate the single term involving $H_j(y)$:

$$\left.\frac{d^j S(y,s)}{ds^j}\right|_{s=0} = j!\frac{H_j(y)}{j!} \tag{10}$$
$$= H_j(y) \tag{11}$$

This is the reason that $S(y,s)$ is called a generating function: it provides a relatively simple way of generating all the Hermite polynomials.

Since we started by defining the generating function, we should prove that the polynomials that it generates really are solutions of Hermite's differential equation. We can do this by taking derivatives of the generating function (but without the step of setting $s=0$). We take derivatives of 4 and 9 and then set them equal to each other.

$$\frac{\partial S}{\partial y} = 2se^{-s^2+2sy} \tag{12}$$
$$= \sum_{m=0}^{\infty}\frac{2s^{m+1}}{m!}H_m(y)\,\text{from }(1) \tag{13}$$
$$\frac{\partial S}{\partial y} = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{dH_m}{dy}s^m\,\text{from }(2) \tag{14}$$

Now we use the old trick of requiring these two results to be equal for all values of s, which implies that the coefficients of each power of s must be equal independently. That is

$$\sum_{m=0}^{\infty}\frac{2s^{m+1}}{m!}H_m(y) = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{dH_m}{dy}s^m \tag{15}$$
$$\sum_{m=1}^{\infty}\frac{2s^{m}}{(m-1)!}H_{m-1}(y) = \sum_{m=1}^{\infty}\frac{1}{m!}\frac{dH_m}{dy}s^m \tag{16}$$

In the second line, we adjusted the summation index on the left so that the power of s was s^m. On the right, we dropped the $m=0$ term since $dH_0/dy=0$ anyway (since $H_0=1$). The two sums are now aligned, so we can say

$$\frac{2}{(m-1)!}H_{m-1}(y) = \frac{1}{m!}\frac{dH_m}{dy} \tag{17}$$
$$2mH_{m-1} = H_m' \tag{18}$$

By a similar process we can take the other derivative with respect to s:

$$\frac{\partial S}{\partial s} = (-2s+2y)e^{-s^2+2sy} \tag{19}$$

$$= \sum_{m=0}^{\infty} \frac{(-2s+2y)s^m}{m!} H_m \text{ from } (1) \tag{20}$$

$$\frac{\partial S}{\partial s} = \sum_{m=1}^{\infty} \frac{m}{m!} H_m s^{m-1} \text{ from } (2) \tag{21}$$

We have ignored the $m=0$ term in the last line, since the derivative of the first term in the series with respect to s is zero. Aligning the powers of s gives

$$-\sum_{m=1}^{\infty} \frac{2s^m}{(m-1)!} H_{m-1} + \sum_{m=0}^{\infty} \frac{2ys^m}{m!} H_m = \sum_{m=0}^{\infty} \frac{1}{m!} H_{m+1} s^m \tag{22}$$

$$-2mH_{m-1} + 2yH_m = H_{m+1} \tag{23}$$

This relation is valid for all m even though the $m=0$ case is a bit fortuitous. With $m=0$ we get $2yH_0 = H_1$ which is true, since $H_0 = 1$ and $H_1 = 2y$.

From these results we can show that the polynomials do in fact solve Hermite's differential equation. We do this by showing that the results above allow us to reconstruct the equation. From the second result:

$$H_{m+1} = 2yH_m - 2mH_{n-1} \tag{24}$$

$$H'_{m+1} = 2H_m + 2yH'_m - 2mH'_{n-1} \tag{25}$$

From the first result, $2mH'_{m-1} = H''_m$, and $H'_{m+1} = 2(m+1)H_m$ so substituting these into the last line above, we get

$$H''_m - 2yH'_m + 2mH_m = 0 \tag{26}$$

which is Hermite's equation. QED.

One final bit of business is to show that the generating function approach is equivalent to the other definition of Hermite polynomials, that is, that it is equivalent to saying

$$H_n \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{27}$$

The generating function 4 can be written as

$$S(y,s) \equiv e^{-s^2+2sy} \tag{28}$$

$$= e^{y^2-(s-y)^2} \tag{29}$$

so taking the derivative, we get

$$\begin{aligned}\frac{\partial^n S}{\partial s^n} &= e^{y^2}\frac{\partial^n}{\partial s^n}e^{-(s-y)^2} && (30)\\ &= (-1)^n e^{y^2}\frac{\partial^n}{\partial y^n}e^{-(s-y)^2} && (31)\end{aligned}$$

since for any function $f(s-y)$, $\partial f/\partial s=-\partial f/\partial y$. Setting $s=0$, we reclaim the original definition:

$$\left.\frac{\partial^n S}{\partial s^n}\right|_{s=0}=(-1)^n e^{y^2}\frac{\partial^n}{\partial y^n}e^{-y^2} \tag{32}$$

so the two definitions are equivalent.

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