HERMITIAN MATRIX THEOREMS

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A *hermitian matrix* A is a matrix that is equal to its hermitian conjugate, that is

$$A^\dagger = A \tag{1}$$

Here, we'll present a few results involving hermitian matrices and hermitian conjugates.

Theorem 1. *For an $n \times n$ hermitian matrix, all its diagonal entries are real.*

Proof. From the definition 1, we must have

$$a_{ij} = \overline{a}_{ji} \tag{2}$$

Thus for a diagonal element $a_{ii} = \overline{a}_{ii}$ so it's equal to its complex conjugate, and thus must be real. □

Theorem 2. *Suppose A is an $n \times n$ hermitian matrix and u is an $n \times 1$ column vector with complex entries. Then $u^\dagger A u$ is a real number.*

Proof. We use an approach similar to that in the previous post. We consider $n = 3$, so that

$$u = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \tag{3}$$

$$u^\dagger = \begin{bmatrix} \overline{u}_1 & \overline{u}_2 & 0 \end{bmatrix} \tag{4}$$

and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{5}$$

Then

$$u^\dagger A = \begin{bmatrix} (\overline{u}_1 a_{11} + \overline{u}_2 a_{21}) & (\overline{u}_1 a_{12} + \overline{u}_2 a_{22}) & (\overline{u}_1 a_{13} + \overline{u}_2 a_{23}) \end{bmatrix} \tag{6}$$

The total product is then

$$u^{\dagger}Au = u_1\overline{u}_1a_{11} + u_1\overline{u}_2a_{21} + u_2\overline{u}_1a_{12} + u_2\overline{u}_2a_{22} \tag{7}$$

$$= |u_1|^2 a_{11} + |u_2|^2 a_{22} + u_1\overline{u}_2a_{21} + u_2\overline{u}_1a_{12} \tag{8}$$

Since a_{11} and a_{22} are diagonal elements and are therefore real, the first two terms are real. From 2, $a_{12} = \overline{a}_{21}$, so the third term is the complex conjugate of the fourth term. Thus the sum of the third and fourth terms is real, thus $u^{\dagger}Au$ is real. The argument can be generalized to arbitrary n. The terms in the sum will all consist either of diagonal elements or sums of complex conjugate pairs. □

Theorem 3. *If B is any $m \times n$ matrix with complex entries, then $B^{\dagger}B$ is hermitian.*

Proof. This follows from the condition of taking the transpose of a product of matrices. For any two matrices C and D (where the number of columns in C equals the number of rows in D so we can form a matrix product)

$$(CD)^T = D^T C^T \tag{9}$$

Forming the hermitian conjugate of a product requires taking the transpose of the product followed by complex conjugation, so we have

$$\left(B^{\dagger}B\right)^{\dagger} = \left[B^{\dagger}B\right]^{T*} \tag{10}$$

where the * indicates complex conjugate. Thus we have

$$\left[B^{\dagger}B\right]^{T*} = \left(B^T B^{\dagger T}\right)^* = B^{\dagger}B \tag{11}$$

since taking the complex conjugate of the transpose of a hermitian conjugate just restores the original matrix. □

Theorem 4. *We now take B to be an $n \times n$ matrix with complex entries and u to be an $n \times 1$ column vector, also with complex entries. Then $u^{\dagger}B^{\dagger}Bu$ is a non-negative real number.*

Proof. Let

$$v \equiv Bu \tag{12}$$

so that v is an $n \times 1$ column vector with complex entries v_i. Then $v^{\dagger} = u^{\dagger}B^{\dagger}$ is a $1 \times n$ row vector whose elements are the complex conjugates of v_i. That is, $\left[v^{\dagger}\right]_i = \overline{v}_i$. Therefore

$$u^\dagger B^\dagger B u = v^\dagger v \tag{13}$$

$$= \sum_{i=1}^{n} \overline{v}_i v_i \tag{14}$$

$$= \sum_{i=1}^{n} |v_i|^2 \geq 0 \tag{15}$$

□