HERMITIAN OPERATORS - A FEW THEOREMS

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We'll look at a few simple theorems about hermitian operators.

Theorem 1. *The sum of two hermitian operators is also hermitian.*

Proof. From the definition:

$$\left\langle f|(\hat{Q}+\hat{R})f\right\rangle = \left\langle f|\hat{Q}f\right\rangle + \left\langle f|\hat{R}f\right\rangle \tag{1}$$

$$= \left\langle \hat{Q}f|f\right\rangle + \left\langle \hat{R}f|f\right\rangle \tag{2}$$

$$= \left\langle (\hat{Q}+\hat{R})f|f\right\rangle \tag{3}$$

□

Theorem 2. *If $\hat{Q}$ is hermitian, then $\alpha\hat{Q}$ is hermitian if α is real.*

Proof. First we note:

$$\left\langle f|\alpha\hat{Q}f\right\rangle = \alpha\left\langle f|\hat{Q}f\right\rangle = \alpha\left\langle \hat{Q}f|f\right\rangle \tag{4}$$

Next, for $\alpha\hat{Q}$ to be hermitian:

$$\left\langle \alpha\hat{Q}f|f\right\rangle = \alpha^*\left\langle \hat{Q}f|f\right\rangle \tag{5}$$

$$= \alpha\left\langle \hat{Q}f|f\right\rangle \tag{6}$$

so we must have $\alpha = \alpha^*$. □

Theorem 3. *The product of two hermitian operators $\hat{Q}$ and $\hat{R}$ is hermitian if the operators commute.*

Proof. We have:

$$\left\langle f|\hat{Q}(\hat{R}f)\right\rangle = \left\langle \hat{Q}f|\hat{R}f\right\rangle \tag{7}$$

$$= \left\langle \hat{R}(\hat{Q}f)|f\right\rangle \tag{8}$$

In order for this to be hermitian, we must have

$$\left\langle \hat{R}(\hat{Q}f)|f\right\rangle = \left\langle \hat{Q}(\hat{R}f)|f\right\rangle \tag{9}$$

so the operators must commute. □

Theorem 4. *The position operator $\hat{x}$ is hermitian.*

Proof. Since $\hat{x}$ is real and its operation is simply multiplication, it doesn't matter where it appears in the inner product:

$$\langle f|\hat{x}f\rangle = \langle \hat{x}f|f\rangle \tag{10}$$

□

Theorem 5. *The hamiltonian is hermitian.*

Proof. To test the hamiltonian operator $H = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$, we note that the potential part $V(x)$ is a real, multiplicative operator so is hermitian by the same reasoning as for $\hat{x}$. The kinetic part consists of a real constant $-\hbar^2/2m$ multiplied by a second derivative, so by Theorem 2 we need show only that the second derivative is hermitian. We can do this by introducing a test function f for the derivative to operate on, and then integrating by parts twice:

$$\langle f|f''\rangle = \int f^* f'' dx \tag{11}$$

$$= f^* f' - \int (f^*)' f' dx \tag{12}$$

$$= f^* f' - (f^*)' f + \int (f^*)'' f dx \tag{13}$$

$$= f^* f' - (f^*)' f + \langle f''|f\rangle \tag{14}$$

We can throw away the two terms $f^* f' - f^{*'} f$ under the usual assumption that all physical functions tend to zero at the boundaries, so we are left with

$$\langle f|f''\rangle = \langle f''|f\rangle \tag{15}$$

The hamiltonian is thus the sum of two hermitian operators, so is itself hermitian by Theorem 1. □