HERMITIAN OPERATORS - EQUIVALENCE OF CONDITIONS

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A Hermitian operator Q satisfies the condition that for inner products we have

$$\langle g|\hat{Q}f\rangle=\langle\hat{Q}g|f\rangle \tag{1}$$

It is also possible to arrive at this result if we start off with the apparently less restrictive condition

$$\langle h|\hat{Q}h\rangle=\langle\hat{Q}h|h\rangle \tag{2}$$

where h is any function in Hilbert space.

To show this, we start with $h=f+g$:

$$\langle h|\hat{Q}h\rangle = \langle f+g|\hat{Q}f+\hat{Q}g\rangle \tag{3}$$
$$= \langle f|\hat{Q}f\rangle+\langle g|\hat{Q}g\rangle+\langle f|\hat{Q}g\rangle+\langle g|\hat{Q}f\rangle \tag{4}$$

But we are assuming that $\langle h|\hat{Q}h\rangle=\langle\hat{Q}h|h\rangle$ so we can write this equation as:

$$\langle\hat{Q}h|h\rangle = \langle\hat{Q}f+\hat{Q}g|f+g\rangle \tag{5}$$
$$= \langle\hat{Q}f|f\rangle+\langle\hat{Q}g|g\rangle+\langle\hat{Q}f|g\rangle+\langle\hat{Q}g|f\rangle \tag{6}$$

Using the assumption $\langle h|\hat{Q}h\rangle=\langle\hat{Q}h|h\rangle$ for all functions, we have $\langle f|\hat{Q}f\rangle=\langle\hat{Q}f|f\rangle$ and $\langle g|\hat{Q}g\rangle=\langle\hat{Q}g|g\rangle$ so, equating the two expansions above and cancelling terms, we get

$$\langle f|\hat{Q}g\rangle+\langle g|\hat{Q}f\rangle=\langle\hat{Q}f|g\rangle+\langle\hat{Q}g|f\rangle \tag{7}$$

Repeating this procedure for $h=f+ig$, we have:

$$\langle h|\hat{Q}h\rangle = \langle f+ig|\hat{Q}f+i\hat{Q}g\rangle \tag{8}$$
$$= \langle f|\hat{Q}f\rangle+(-i)(i)\langle g|\hat{Q}g\rangle+i\langle f|\hat{Q}g\rangle-i\langle g|\hat{Q}f\rangle \tag{9}$$
$$= \langle f|\hat{Q}f\rangle+\langle g|\hat{Q}g\rangle+i\langle f|\hat{Q}g\rangle-i\langle g|\hat{Q}f\rangle \tag{10}$$

Here, the factor of $-i$ in the second term in the second line comes from the complex conjugate of ig.

Finally, we calculate $\langle\hat{Q}h|h\rangle$ for $h = f + ig$ to get

$$\langle\hat{Q}h|h\rangle = \langle\hat{Q}f|f\rangle + \langle\hat{Q}g|g\rangle + i\langle\hat{Q}f|g\rangle - i\langle\hat{Q}g|f\rangle \tag{11}$$

(The $+i$ and $-i$ multiply to give 1 in the second term $\langle\hat{Q}g|g\rangle$.)

Equating the last two expansions, as before, leads to:

$$\langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle\hat{Q}f|g\rangle - \langle\hat{Q}g|f\rangle \tag{12}$$

If we now add equations 7 and 12 we get:

$$\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle \tag{13}$$

Similarly if we subtract 12 from 7 we get:

$$\langle g|\hat{Q}f\rangle = \langle\hat{Q}g|f\rangle \tag{14}$$

Thus we reclaim the more general Hermitian condition we started with.